Integral models in unramified mixed characteristic (0,2) of hermitian orthogonal Shimura varieties of PEL type, Part II

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ABSTRACT. We construct relative PEL type embeddings in mixed characteristic $(0,2)$ between hermitian orthogonal Shimura varieties of PEL type. We use this to prove the existence of integral canonical models in unramified mixed characteristic $(0,2)$ of hermitian orthogonal Shimura varieties of PEL type.

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Contents

1. Introduction

This paper is a sequel to [19] and thus its main goal is to prove (unconditionally) the existence of *integral canonical models* in unramified mixed characteristic $(0,2)$ of *hermitian orthogonal Shimura varieties of PEL type*. We begin with a review on Shimura varieties (of Hodge and PEL type). Let $\mathbf{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_{m\mathbf{C}}$ be the unique two dimensional torus over $\mathbf{R}$ with the property that $\mathbf{S}(\mathbf{R})$ is the (multiplicative) group $\mathbf{G}_{m\mathbf{C}}(\mathbf{C})$ of non-zero complex numbers. A Shimura pair $(G,\mathcal{X})$ comprises from a reductive group G over $\mathbf{Q}$ and a $G(\mathbf{R})$-conjugacy class $\mathcal{X}$ of homomorphisms $\mathbf{S} \to G_{\mathbf{R}}$ that satisfy Deligne's axioms of [4, Subsubsection 2.1.1]: (i) the Hodge $\mathbf{Q}$–structure on the Lie algebra $\mathrm{Lie}(G)$ of G defined by any element $h \in \mathcal{X}$ is of type $\{(-1,1),(0,0),(1,-1)\}$, (ii) no simple factor of the adjoint group G^{ad} of G becomes compact over $\mathbf{R}$, and (iii) $\mathrm{Ad} \circ h(i)$ is a Cartan involution of $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbf{R}})$ in the sense of [9, Ch. III, §7]. Here $\mathrm{Ad} : G_{\mathbf{R}} \to \mathbf{GL}_{\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbf{R}})}$ is the adjoint representation. The axioms imply that $\mathcal{X}$ has a natural structure of a hermitian symmetric domain, cf. [4, Cor. 1.1.17].

The most studied Shimura pairs are of the form $(\mathbf{GSp}(W,\psi),\mathcal{S})$, where (W,ψ) is a symplectic space over $\mathbf{Q}$ and where $\mathcal{S}$ is the set of all $\mathbf{R}$-monomorphisms $\mathbf{S}\hookrightarrow \mathbf{GSp}(W,\psi)_{\mathbf{R}}$ that define Hodge $\mathbf{Q}$–structures on W of type $\{(-1,0),(0,-1)\}$ and that have either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. We assume that the Shimura pair $(G,\mathcal{X})$ is of *Hodge type* i.e., there exists an injective map

$$f:(G,\mathcal{X})\hookrightarrow(\mathbf{GSp}(W,\psi),\mathcal{S})$$

of Shimura pairs (see [3], [4], [11, Ch. 1], [12], and [17, Subsection 2.4]). To $(G,\mathcal{X})$ it is associated a number field $E(G,\mathcal{X})$ called the *reflex field* (see [3], [4], and [11]) as well as a canonical model $\mathrm{Sh}(G,\mathcal{X})$ over $E(G,\mathcal{X})$ (see [14] and [3]).

Let $\mathbf{Z}_{(2)}$ be the localization of $\mathbf{Z}$ with respect to the prime 2. Let L be a $\mathbf{Z}$-lattice of W such that ψ induces a perfect form $\psi: L\otimes_{\mathbf{Z}} L\to\mathbf{Z}$ i.e., the induced monomorphism $L\hookrightarrow L^*:=\mathrm{Hom}(L,\mathbf{Z})$ is onto. Let $L_{(2)}:=L\otimes_{\mathbf{Z}}\mathbf{Z}_{(2)}$. Let $G_{\mathbf{Z}_{(2)}}$ be the Zariski closure of G in the reductive group scheme $\mathbf{GSp}(L_{(2)},\psi)$. Let $K_2:=\mathbf{GSp}(L_{(2)},\psi)(\mathbf{Z}_2)$ and $H_2:=G(\mathbf{Q}_2)\cap K_2=G_{\mathbf{Z}_{(2)}}(\mathbf{Z}_2)$. Let

$$\mathcal{B}:=\{b\in\mathrm{End}(L_{(2)})|b \text{ is fixed by } G_{\mathbf{Z}_{(2)}}\}.$$

Let G_1 be the subgroup of $\mathbf{GSp}(W,\psi)$ that fixes all elements of $\mathcal{B}[\frac{1}{2}]$.

Let $\mathcal{I}$ be the involution of $\mathrm{End}(L_{(2)})$ defined by the identity $\psi(b(l_1),l_2)=\psi(l_1,\mathcal{I}(b)(l_2))$, where $b\in\mathrm{End}(L_{(2)})$ and $l_1,\, l_2\in L_{(2)}$. As $\mathcal{B}=\mathcal{B}[\frac{1}{2}]\cap\mathrm{End}(L_{(2)})$, we have $\mathcal{I}(\mathcal{B})=\mathcal{B}$. As the elements of $\mathcal{X}$ fix $\mathcal{B}\otimes_{\mathbf{Z}_{(2)}}\mathbf{R}$, the involution $\mathcal{I}$ of $\mathcal{B}$ is positive.

Let $\mathbf{F}$ be an algebraic closure of the field $\mathbf{F}_2$ with two elements. Let $W(\mathbf{F})$ be the ring of Witt vectors with coefficients in $\mathbf{F}$. Let $\mathbf{A}_f:=\widehat{\mathbf{Z}}\otimes_{\mathbf{Z}}\mathbf{Q}$ be the ring of finite adèles. Let $\mathbf{A}_f^{(2)}$ be the ring of finite adèles with the 2-component omitted; we have $\mathbf{A}_f=\mathbf{Q}_2\times\mathbf{A}_f^{(2)}$. Let v be a prime of $E(G,\mathcal{X})$ that divides 2. Let $O_{(v)}$ be the localization of the ring of integers of $E(G,\mathcal{X})$ with respect to the prime v.

1.1. Shimura pairs of PEL type. In this paper we are interested in Shimura pairs of *PEL type*. Thus we will assume that the following four properties (axioms) hold:

(i) the $W(\mathbf{F})$-algebra $\mathcal{B}\otimes_{\mathbf{Z}_{(2)}}W(\mathbf{F})$ is a product of matrix $W(\mathbf{F})$-algebras;

(ii) the $\mathbf{Q}$–algebra $\mathcal{B}[\frac{1}{2}]$ is $\mathbf{Q}$–simple;

(iii) the group G is the identity component of G_1;

(iv) the flat, affine group scheme $G_{\mathbf{Z}_{(2)}}$ over $\mathbf{Z}_{(2)}$ is reductive (i.e., it is smooth and its special fibre is connected and has a trivial unipotent radical).

Always (iv) implies (i) but this is irrelevant in what follows. Assumption (iv) implies that H_2 is a hyperspecial subgroup of $G(\mathbf{Q}_2)=G_{\mathbf{Q}_2}(\mathbf{Q}_2)$ (cf. [16, Subsubsection 3.8.1]) and that the prime v is unramified over 2 (cf. [12, Cor. 4.7 (a)]). Let G^{der} be the derived group of G. Assumption (ii) is not really required: it is inserted only to ease the presentation. Due to the properties (ii) and (iii), one distinguishes the following three possible (and disjoint) cases (see [10, §7]):

(A) the group $G^{\mathrm{der}}_{\mathbf{C}}$ is a product of $\mathbf{SL}_n$ groups with $n \geq 2$ and, in the case $n = 2$, the center of G has dimension at least 2;

(C) the group $G^{\mathrm{der}}_{\mathbf{C}}$ is a product of $\mathbf{Sp}_{2n}$ groups with $n \geq 1$ and, in the case $n = 1$, the center of G has dimension 1;

(D) the group $G^{\mathrm{der}}_{\mathbf{C}}$ is a product of $\mathbf{SO}_{2n}$ groups with $n \geq 2$.

We have $G \neq G_1$ if and only if we are in the case (D) i.e., if and only if G^{der} is not simply connected (cf. [10, §7]). We recall that PEL stands for polarization, endomorphisms, and level structures and that the first two of these notions refer to the fact that the axiom (iii) holds. In the case (A) (resp. (C) or (D)), one often says that $\mathrm{Sh}(G, \mathcal{X})$ is a *unitary* (resp. *symplectic* or *hermitian orthogonal*) Shimura variety of PEL type (cf. the description of the intersection group $G_{\mathbf{R}} \cap \mathbf{Sp}(W \otimes_{\mathbf{Q}} \mathbf{R}, \psi)$ in [14, Subsections 2.6 and 2.7]). We are in the case (D) if and only if $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} \mathbf{R}$ is a product of matrix algebras over the quaternion $\mathbf{R}$-algebra $\mathbf{H}$ (see [14, Subsection 2.1, (type III)]).

In the cases (A) and (C), $G_{\mathbf{Z}_{(2)}}$ is the subgroup scheme of $\mathbf{GSp}(L_{(2)}, \psi)$ that fixes $\mathcal{B}$. But in the case (D) we have the following serious problem: the subgroup scheme of $\mathbf{GSp}(L_{(2)}, \psi)$ that fixes $\mathcal{B}$ is not smooth and its identity component is not $G_{\mathbf{Z}_{(2)}}$ (see [19, Subsubsection 3.5.1]). For the rest of the paper we also assume that:

(v) we are in the case (D).

We refer to each quadruple $(f, L, v, \mathcal{B})$ that satisfies properties (i) to (v) as a *hermitian orthogonal standard PEL situation* in mixed characteristic $(0, 2)$.

Let $\mathcal{M}$ be the $\mathbf{Z}_{(2)}$-scheme which parameterizes isomorphism classes of principally polarized abelian schemes that are of relative dimension $\frac{\dim_{\mathbf{Q}}(W)}{2}$ over $\mathbf{Z}_{(2)}$-schemes and that are equipped with compatible level-l symplectic similitude structures for all odd numbers $l \in \mathbf{N}$, cf. [13, Thms. 7.9 and 7.10]. We have a natural identification $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{E(G,\mathcal{X})}/K_2 = \mathcal{M}_{E(G,\mathcal{X})}$ as well as an action of $\mathbf{GSp}(W, \psi)(\mathbf{A}_f^{(2)})$ on $\mathcal{M}$. These symplectic similitude structures and this action are defined naturally via (L, ψ) (see [3, Example 4.16], [12, Section 3], and [17, Subsection 4.1]).

It is well known that we have and identity $\mathrm{Sh}(G, \mathcal{X})_{\mathbf{C}}/H_2 = G_{\mathbf{Z}_{(2)}}(\mathbf{Z}_{(2)}) \backslash \mathcal{X} \times G(\mathbf{A}_f^{(2)})$ and that to the injective map f one associates a natural $E(G, \mathcal{X})$-morphism

$$\mathrm{Sh}(G, \mathcal{X})/H_2 \to \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{S})_{E(G,\mathcal{X})}/K_2 = \mathcal{M}_{E(G,\mathcal{X})}$$

which is a closed embedding (for instance, see [19, Subsection 1.3]).

Let $\mathcal{N}$ be the Zariski closure of $\mathrm{Sh}(G, \mathcal{X})/H_2$ in $\mathcal{M}_{O_{(v)}}$. Let $\mathcal{N}^{\mathrm{n}}$ be the normalization of $\mathcal{N}$. Let $\mathcal{N}^{\mathrm{s}}$ be the formally smooth locus of $\mathcal{N}^{\mathrm{n}}$ over $O_{(v)}$.

The goal of this Part II is to prove the following two Theorems.

1.2. Basic Theorem. *Let $(f, L, v, \mathcal{B})$ be a hermitian orthogonal standard PEL situation in mixed characteristic $(0, 2)$. Then there exists an embedding $\tilde{f} : (G, \mathcal{X}) \to (\mathbf{GSp}(\tilde{W}, \tilde{\psi}), \tilde{\mathcal{S}})$ of Shimura pairs and a $\mathbf{Z}$-lattice $\tilde{L}$ of $\tilde{W}$ such that $\tilde{f}$ factors through*

an injective map $\tilde{f}' : (G', \mathcal{X}') \hookrightarrow (\mathbf{GSp}(\tilde{W}, \tilde{\psi}), \tilde{\mathcal{S}})$ *in such a way that by defining* $\tilde{L}_{(2)} := \tilde{L} \otimes_{\mathbf{Z}} \mathbf{Z}_{(2)}$ *the following three properties hold:*

(i) *the Zariski closure of* G *in* $\mathbf{GL}_{\tilde{L}_{(2)}}$ *is* $G_{\mathbf{Z}_{(2)}}$ *and the Zariski closure* $G'_{\mathbf{Z}_{(2)}}$ *of* G' *in* $\mathbf{GL}_{\tilde{L}_{(2)}}$ *is a reductive group scheme over* $\mathbf{Z}_{(2)}$ *whose extension to* $\mathbf{Z}_2$ *is a split* $\mathbf{GSO}_{2nq}$ *group scheme for some* $q \in \mathbf{N}$*;*

(ii) *if* $\tilde{\mathcal{B}} := \{b \in \mathrm{End}(\tilde{L}_{(2)}) | b \text{ is fixed by } G_{\mathbf{Z}_{(2)}}\}$*,* v' *is the prime of* $E(G', \mathcal{X}')$ *divided naturally by* v*, and* $\tilde{\mathcal{B}} := \{b \in \mathrm{End}(\tilde{L}_{(2)}) | b \text{ is fixed by } G'_{\mathbf{Z}_{(2)}}\}$*, then both quadruples* $(\tilde{f}, \tilde{L}, v, \tilde{\mathcal{B}})$ *and* $(\tilde{f}', \tilde{L}, v', \tilde{\mathcal{B}}')$ *are hermitian orthogonal standard PEL situations in mixed characteristic* $(0,2)$*;*

(iii) *the identity component of the subgroup scheme of* $G'_{\mathbf{Z}_{(2)}}$ *that fixes* $\tilde{\mathcal{B}}$ *is* $G_{\mathbf{Z}_{(2)}}$*.*

We refer to the quadruple $(\tilde{f}, \tilde{f}', \tilde{L}, v)$ as a *hermitian orthogonal relative PEL situation*; such a relative situation is similar to (though different from) the relative PEL situations used in [17, Subsubsection 4.3.16 and §6]. Property (iii) surpasses the serious problem we mentioned in Subsection 1.1. In [19] we proved that if $G_{\mathbf{Z}_2}$ is a split $\mathbf{GSO}_{2n}$ group scheme, then we have $\mathcal{N}^{\mathrm{n}} = \mathcal{N}^{\mathrm{s}}$. By combining this result with the Basic Theorem we prove (unconditionally) that:

1.3. Main Theorem. *Let* $(f, L, v, \mathcal{B})$ *be a hermitian orthogonal standard PEL situation in mixed characteristic* $(0,2)$*. Let* $\mathcal{N}^{\mathrm{s}}$ *and* $\mathcal{N}^{\mathrm{m}}$ *be the* $O_{(v)}$*-schemes obtained as in Subsection 1.1. Then we have an identity* $\mathcal{N}^{\mathrm{s}} = \mathcal{N}^{\mathrm{n}}$ *i.e., the* $O_{(v)}$*-scheme* $\mathcal{N}^{\mathrm{n}}$ *is regular and formally smooth.*

The locally compact, totally disconnected topological group $G(\mathbf{A}_f^{(2)})$ acts on $\mathcal{N}^{\mathrm{n}}$ continuously in the sense of [4, Subsubsection 2.7.1]. Thus $\mathcal{N}^{\mathrm{n}}$ is an integral canonical model of $\mathrm{Sh}(G, \mathcal{X})/H_2$ over $O_{(v)}$ in the sense of [17, Def. 3.2.3 6)], cf. [17, Example 3.2.9 and Cor. 3.4.4]. Due to [18, Thm. 1.3], as in [17, Rms. 3.2.4 and 3.2.7 4')] we get that $\mathcal{N}^{\mathrm{n}}$ is the *unique integral canonical model* of $\mathrm{Sh}(G, \mathcal{X})/H_2$ over $O_{(v)}$ and that $\mathcal{N}^{\mathrm{n}}$ is the *final object* of the category of smooth integral models of $\mathrm{Sh}(G, \mathcal{X})/H_2$ over $O_{(v)}$ (here the word smooth is used as in [11, Def. 2.2]).

In Section 2 we present tools that pertain to group schemes and that are needed to prove the Basic Theorem in Section 3. In Section 4 we recall few basic crystalline properties from [19]. In Section 5 we prove the Main Theorem.

2. Group schemes

Let $n \in \mathbf{N}$. Let $\mathrm{Spec}(S)$ be an affine scheme. We recall that a reductive group scheme $\mathcal{R}$ over S is a smooth, affine group scheme over S whose fibres are connected and have trivial unipotent radicals. Let $\mathcal{R}^{\mathrm{ad}}$ and $\mathcal{R}^{\mathrm{der}}$ be the adjoint and the derived (respectively) group schemes of $\mathcal{R}$, cf. [6, Vol. III, Exp. XXII, Def. 4.3.6 and Thm. 6.2.1]. Let $\mathrm{Lie}(\mathcal{U})$ be the Lie algebra of a smooth, closed subgroup scheme $\mathcal{U}$ of $\mathcal{R}$. For an affine morphism $\mathrm{Spec}(S_1) \to \mathrm{Spec}(S)$ and for Z (or Z_S or Z_*) an S-scheme, let Z_{S_1}

(or Z_{S_1} or Z_{*S_1}) be $Z \times_S S_1$. If $\tilde{S} \hookrightarrow S$ is a finite, étale $\mathbf{Z}$-monomorphism, let $\mathrm{Res}_{S/\tilde{S}}$ be the operation of Weil restriction from S to $\tilde{S}$ (see [1, Ch. 7, 7.6]). Thus $\mathrm{Res}_{S/\tilde{S}}\mathcal{R}$ is a reductive group scheme over $\tilde{S}$ such that for each $\tilde{S}$-scheme Y we have a functorial group identification $\mathrm{Res}_{S/\tilde{S}}\mathcal{R}(Y) = \mathcal{R}(Y \times_S \tilde{S})$.

If M is a free S-module of finite rank, let $M^* := \mathrm{Hom}(M, S)$, let $\mathbf{GL}_M$ be the reductive group scheme over S of linear automorphisms of M, and let $\mathcal{T}(M) := \oplus_{s,t\in\mathbf{N}\cup\{0\}} M^{\otimes s} \otimes_S M^{*\otimes t}$. Each S-linear isomorphism $i : M \tilde{\to} \tilde{M}$ of free S-modules of finite rank, extends naturally to an S-linear isomorphism (to be denoted also by) $i : \mathcal{T}(M) \tilde{\to} \mathcal{T}(\tilde{M})$ and therefore we will speak about i taking some tensor of $\mathcal{T}(M)$ to some tensor of $\mathcal{T}(\tilde{M})$. We identify $\mathrm{End}(M) = M \otimes_S M^*$. A bilinear form λ_M on a free S-module M of finite rank is called perfect if it induces an S-linear isomorphism $M \tilde{\to} M^*$. If λ_M is alternating, we call the pair (M, λ_M) a symplectic space over S and we define $\mathbf{Sp}(M, \lambda_M) := \mathbf{GSp}(M, \lambda_M)^{\mathrm{der}}$. We often use the same notation for two elements of some modules (like involutions, endomorphisms, bilinear forms, etc.) that are obtained one from another via extensions of scalars and restrictions. If E (or E_*) is a number field, let $E_{(2)}$ (or $E_{*(2)}$) be the normalization of $\mathbf{Z}_{(2)}$ in E (or E_*).

The reductive group schemes $\mathbf{Sp}_{2nS}$, $\mathbf{GL}_{nS}$, etc., are over S. The Lie groups $\mathbf{U}(n)$, $\mathbf{Sp}(n, \mathbf{R})$, $\mathbf{SL}(n, \mathbf{C})$, $\mathbf{SO}^*(2n)$, etc., are as in [9, Ch. X, §2, 1] (but using boldface capital letters). Let $\mathbf{SO}^+_{2nS}$, $\mathbf{GSO}^+_{2nS}$, and $\mathbf{O}^+_{2nS}$ be the split $\mathbf{SO}_{2n}$, $\mathbf{GSO}_{2n}$, and $\mathbf{O}_{2n}$ (respectively) reductive group schemes over S. We recall that $\mathbf{GSO}^+_{2nS}$ is the quotient of $\mathbf{SO}^+_{2nS} \times_S \mathbf{G}_{mS}$ by a $\boldsymbol{\mu}_{2S}$ subgroup scheme that is embedded diagonally.

In Subsection 2.1 we review some general facts on $\mathbf{SO}^+_{2n\mathbf{Z}_{(2)}}$. In Subsection 2.2 we study hermitian twists of split $\mathbf{SO}$ group schemes and the Lie groups associated to them.

2.1. Split $\mathbf{SO}_{2n}$ group schemes. We consider the quadratic form

$$\mathfrak{Q}_n(x) := x_1x_2 + \cdots + x_{2n-1}x_{2n} \quad \text{defined for} \quad x = (x_1, \ldots, x_{2n}) \in \mathcal{L}_n := \mathbf{Z}^{2n}_{(2)}.$$

For $\alpha \in \mathbf{Z}_{(2)}$ and $x \in \mathcal{L}_n$ we have $\mathfrak{Q}_n(\alpha x) = \alpha^2\mathfrak{Q}_n(x)$. Let $\tilde{\mathcal{D}}_n$ be the subgroup scheme of $\mathbf{GL}_{\mathcal{L}_n}$ that fixes $\mathfrak{Q}_n$. Let $\mathcal{D}_n$ be the Zariski closure of the identity component of $\tilde{\mathcal{D}}_{n\mathbf{Q}}$ in $\tilde{\mathcal{D}}_n$. We recall from [19, Subsection 3.1] that $\mathcal{D}_n$ and $\tilde{\mathcal{D}}_n$ are isomorphic to $\mathbf{SO}^+_{2n\mathbf{Z}_{(2)}}$ and $\mathbf{O}^+_{2n\mathbf{Z}_{(2)}}$ (respectively). Thus $\mathcal{D}_1$ is isomorphic to $\mathbf{G}_{m\mathbf{Z}_{(2)}}$, $\mathcal{D}_2$ is isomorphic to the quotient of a product of two copies of $\mathbf{SL}_{2\mathbf{Z}_{(2)}}$ by a $\boldsymbol{\mu}_{2\mathbf{Z}_{(2)}}$ subgroup scheme that is embedded diagonally, and for $n \geq 2$ the group scheme $\mathcal{D}_n$ is semisimple. Moreover we have a non-trivial, split short exact sequence $0 \to \mathcal{D}_n \to \tilde{\mathcal{D}}_n \to \mathbf{Z}/2\mathbf{Z}_{\mathbf{Z}_{(2)}} \to 0$ and $\mathcal{D}_n$ is the identity component of $\tilde{\mathcal{D}}_n$. Let

$$\rho_n : \mathcal{D}_n \hookrightarrow \mathbf{GL}_{\mathcal{L}_n}$$

be the natural rank $2n$ faithful representation. We also recall from [19, Subsection 3.1] that ρ_n is associated to the weight ϖ_1 if $n \geq 4$ and to the weight ϖ_2 of the A_3 Lie type if $n = 3$ (see [2, plates I and IV] for these weights); moreover ρ_2 is the tensor product

of the standard rank 2 representations of the mentioned two copies of $\mathbf{SL}_{2\mathbf{Z}_{(2)}}$. Thus the representation ρ_n is isomorphic to its dual and, up to a $\mathbf{G}_m(\mathbf{Z}_{(2)})$-multiple, there exists a unique perfect symmetric bilinear form $\mathfrak{B}_n$ on $\mathcal{L}_n$ fixed by $\mathcal{D}_n$ (the case $n=1$ is trivial). In fact we can take $\mathfrak{B}_n$ such that we have $\mathfrak{B}_n(u,x) := \mathfrak{Q}_n(u+x) - \mathfrak{Q}_n(u) - \mathfrak{Q}_n(x)$ for all $u, x \in \mathcal{L}_n$. Let $J(2n)$ be the matrix representation of $\mathfrak{B}_n$ with respect to the standard $\mathbf{Z}_{(2)}$-basis for $\mathcal{L}_n$; it has n diagonal blocks that are $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Let $q \in \mathbf{N}$. Let

$$d_{n,q} : \mathcal{D}_n^q \hookrightarrow \mathcal{D}_{nq}$$

be a $\mathbf{Z}_{(2)}$-monomorphism such that the faithful representation $\rho_{nq} \circ d_{n,q} : \mathcal{D}_n^q \hookrightarrow \mathbf{GL}_{\mathcal{L}_{nq}}$ is isomorphic to a direct sum of q copies of the representation ρ_n; it is easy to see that $d_{n,q}$ is unique up to $\tilde{\mathcal{D}}_{nq}(\mathbf{Z}_{(2)})$-conjugation.

2.1.1. Lemma. *Let $r \in \mathbf{N}$ be a divisor of q. Let $\Delta_{n,r,q} : \mathcal{D}_n^r \hookrightarrow \mathcal{D}_n^q$ be the natural diagonal embedding. The composite monomorphism $d_{n,q} \circ \Delta_{n,r,q} : \mathcal{D}_n^r \hookrightarrow \mathcal{D}_{nq}$ is a closed embedding that allows us to view $\mathcal{D}_n^r$ as a subgroup scheme of $\mathbf{GL}_{\mathcal{L}_{nq}}$. Let $\mathcal{B}_{n,r,q}$ be the semisimple $\mathbf{Z}_{(2)}$-subalgebra of $\mathrm{End}(\mathcal{L}_{nq})$ formed by endomorphisms of $\mathcal{L}_{nq}$ fixed by $\mathcal{D}_n^r$. Then $\mathcal{D}_n^r$ is the identity component of the centralizer of $\mathcal{B}_{n,r,q}$ in $\mathcal{D}_{nq}$.*

Proof: The centralizer $\mathcal{C}_{n,r,q}$ of $\mathcal{B}_{n,r,q}$ in $\mathbf{GL}_{\mathcal{L}_{nq}}$ is a $\mathbf{GL}^r_{2n\mathbf{Z}_{(2)}}$ group scheme. As $\mathcal{C}_{n,r,q} \cap \mathcal{D}_n^q = \mathcal{D}_n^r$, to prove the Lemma we can assume that $r = q$. We identify $(\mathcal{L}_{nq}, \mathfrak{B}_{nq}) = \oplus_{i=1}^q (\mathcal{L}_n, \mathfrak{B}_n)$ in such a way that the identity $\mathcal{L}_{nq} = \oplus_{i=1}^q \mathcal{L}_n$ defines $d_{n,q}$. Then $\mathcal{C}_{n,q,q} = \prod_{i=1}^n \mathbf{GL}_{\mathcal{L}_n}$ and $\mathcal{C}_{n,q,q} \cap \tilde{\mathcal{D}}_{nq} = \prod_{i=1}^q \tilde{\mathcal{D}}_n = \tilde{\mathcal{D}}_n^q$. Thus the identity component of the centralizer of $\mathcal{B}_{n,q,q}$ in $\mathcal{D}_{nq}$ is the identity component of $\tilde{\mathcal{D}}_n^q$ and therefore it is $\mathcal{D}_n^q$. $\square$

2.2. Twists of $\mathcal{D}_n$. In this Subsection we list few properties of different twists of $\mathcal{D}_n$ and of the Lie groups associated to them.

2.2.1. The $\mathbf{SO}^*(2n)$ Lie group. Let $\mathbf{SO}^*(2n)$ be the Lie group over $\mathbf{R}$ formed by elements of $\mathbf{SL}(2n, \mathbf{C})$ that fix the quadratic form $z_1^2 + \cdots + z_{2n}^2$ as well as the skew hermitian form $-z_1\bar{z}_{n+1} + z_{n+1}\bar{z}_1 - \cdots - z_n\bar{z}_{2n} + z_{2n}\bar{z}_n$. It is connected (cf. [9, Ch. X, §2, 2.4]) and it is associated to a semisimple group over $\mathbf{R}$ that is a form of $\mathcal{D}_{n\mathbf{R}}$.

For $s \in \{1, \ldots, q\}$, let $(z_1^{(s)}, \ldots, z_{2n}^{(s)})$ be variables that define an s-copy of $\mathbf{SO}^*(2n)$. The map that takes the $2nq$-tuple $(z_1^{(1)}, \ldots, z_{2n}^{(1)}, \ldots, z_1^{(q)}, \ldots, z_{2n}^{(q)})$ to the $2nq$-tuple $(z_1^{(1)}, \ldots, z_n^{(1)}, \ldots, z_1^{(q)}, \ldots, z_n^{(q)}, z_{n+1}^{(1)}, \ldots, z_{2n}^{(1)}, \ldots, z_{n+1}^{(q)}, \ldots, z_{2n}^{(q)})$, gives birth to the standard monomorphism of Lie groups

$$e_{n,q} : \mathbf{SO}^*(2n)^q \hookrightarrow \mathbf{SO}^*(2nq).$$

Let

$$s_{n,q} : \mathbf{SO}^*(2n) \hookrightarrow \mathbf{SO}^*(2nq)$$

be the composite of the diagonal embedding $\mathbf{SO}^*(2n) \hookrightarrow \mathbf{SO}^*(2n)^q$ with $e_{n,q}$.

2.2.2. Lemma. *The following three properties hold:*

(a) *we have* $s_{n,q} \circ s_{1,n} = s_{1,nq}$*;*

(b) *the Lie group* $\mathbf{SO}^*(2)$ *is a compact Lie torus of rank* 1*;*

(c) *the centralizer* $C_{1,n}$ *of the image of* $s_{1,n}$ *in* $\mathbf{SO}^*(2n)$ *is a maximal compact Lie subgroup of* $\mathbf{SO}^*(2n)$ *isomorphic to* $\mathbf{U}(n)$*.*

Proof: Part (a) is obvious. It is easy to see that $\mathbf{SO}^*(2)$ is the Lie subgroup of $\mathbf{SL}(2, \mathbf{C})$ whose elements take (z_1, z_2) to $(\cos(\theta)z_1 + \sin(\theta)z_2, -\sin(\theta)z_1 + \cos(\theta)z_2)$ for some $\theta \in \mathbf{R}$. From this (b) follows.

We check (c). Let $u_n : \mathbf{U}(n) \hookrightarrow \mathbf{SO}^*(2n)$ be the Lie monomorphism that takes $X + iY \in \mathbf{U}(n) \leqslant \mathbf{GL}(n, \mathbf{C})$ to $\left(\begin{smallmatrix} X & Y \\ -Y & X \end{smallmatrix}\right) \in \mathbf{SO}^*(2n) \leqslant \mathbf{GL}(2n, \mathbf{C})$, where both X and Y are real $n \times n$ matrices (see [9, Ch. X, §2, 3, Type D III]). The image through u_n of the center of $\mathbf{U}(n)$ is $\mathrm{Im}(s_{1,n})$. Thus we have $\mathrm{Im}(u_n) \leqslant C_{1,n}$. But the centralizer of the image of the complexification of $s_{1,n}$ in the complexification of $\mathbf{SO}^*(2n)$ is a $\mathbf{GL}(n, \mathbf{C})$ Lie group (this follows easily from the definition of $s_{1,n}$). Thus $C_{1,n}$ is a form of $\mathbf{GL}(n, \mathbf{R})$. By reasons of dimensions we get that $C_{1,n} = \mathrm{Im}(u_n) \tilde{\to} \mathbf{U}(n)$. The fact that $C_{1,n}$ is a maximal compact Lie subgroup of $\mathbf{SO}^*(2n)$ follows from [9, Ch. X, §6, Table V]. Thus (c) holds. $\square$

2.2.3. Hermitian twists. Let $\mathrm{Spec}(S_2) \to \mathrm{Spec}(S_1)$ be an étale cover of degree 2 between regular schemes that are flat over $\mathbf{Z}_{(2)}$ and with S_1 integral. There exists a unique S_1-automorphism $\tau \in \mathrm{Aut}_{S_1}(S_2)$ of order 2. Let $M_1 := S_1^{2n}$ and $M_2 := S_2^{2n} = M_1 \otimes_{S_1} S_2$. We view also $\mathfrak{Q}_n(x)$ as a quadratic form defined for $x = (x_1, \ldots, x_{2n}) \in M_2$. Let

$$\mathfrak{H}_n^*(x) := x_1\tau(x_1) - x_2\tau(x_2) + \cdots + x_{2n-1}\tau(x_{2n-1}) - x_{2n}\tau(x_{2n});$$

it is a skew hermitian quadratic form with respect to τ on $M_2 = S_2^{2n}$. Let $\mathcal{D}_{n,\tau}$ be the subgroup scheme of $\mathrm{Res}_{S_2/S_1}\mathcal{D}_{n S_2}$ that fixes $\mathfrak{H}_n^*$.

2.2.4. Lemma. *We have the following four properties:*

(a) *the group scheme* $\mathcal{D}_{n,\tau}$ *over* S_1 *is reductive, splits over* S_2*, and for* $n \geq 2$ *it is semisimple;*

(b) *if* $S_2 = S_1 \oplus S_1$*, then* $\mathcal{D}_{n,\tau}$ *is isomorphic to* $\mathcal{D}_{n S_1}$ *and thus it is split;*

(c) *if* $S_1 \hookrightarrow S_2$ *is* $\mathbf{R} \hookrightarrow \mathbf{C}$ *(thus* τ *is the complex conjugation), then* $\mathcal{D}_{n,\tau}(\mathbf{R})$ *is isomorphic to* $\mathbf{SO}^*(2n)$*;*

(d) *there exist* S_1*-monomorphisms* $\mathcal{D}_{n,\tau} \hookrightarrow \mathbf{Sp}_{8n S_1}$ *such that the resulting rank* $8n$ *representation of* $\mathcal{D}_{n,\tau S_2}$ *is isomorphic to four copies of the representation* $\rho_{n S_2}$*.*

Proof: To prove (a) and (b) we can assume that $S_2 = S_1 \oplus S_1$; thus τ is the permutation of the two factors S_1 of S_2. For $j \in \{1, \ldots, 2n\}$ we write

$$x_j = (u_j, (-1)^{j+1}v_j) \in S_2 = S_1 \oplus S_1.$$

Thus we have identities $\mathfrak{Q}_n(x) = (u_1u_2 + \cdots + u_{2n-1}u_{2n}, v_1v_2 + \cdots + v_{2n-1}v_{2n})$ and $\mathfrak{H}_n^*(x) = (\sum_{j=1}^{2n} u_jv_j, \sum_{j=1}^{2n} u_jv_j)$. Let

$$(g_1, g_2) \in \mathrm{Res}_{S_2/S_1}\mathcal{D}_{n S_2}(S_1) = \mathcal{D}_n(S_1) \times \mathcal{D}_n(S_1) \leqslant \mathbf{GL}_{M_1}(S_1) \times \mathbf{GL}_{M_1}(S_1)$$

be such that it fixes $\sum_{j=1}^{2n} u_j v_j$; here g_1 and g_2 act on an M_1 copy that involves the variables $(u_1, \ldots, u_{2n})$ and $(v_1, \ldots, v_{2n})$ (respectively). We get that at the level of $2n \times 2n$ matrices we have $g_2 = (g_1^t)^{-1} = J(2n) g_1 J(2n)$. Thus the S_1-monomorphism $\mathcal{D}_{n,\tau} \hookrightarrow \mathrm{Res}_{S_2/S_1} \mathcal{D}_{nS_2}$ is isomorphic to the diagonal S_1-monomorphism $\mathcal{D}_{nS_1} \hookrightarrow \mathcal{D}_{nS_1} \times_{S_1} \mathcal{D}_{nS_1}$ and therefore $\mathcal{D}_{n,\tau}$ is isomorphic to $\mathcal{D}_{nS_1}$. From this (a) and (b) follow.

We check (c). Let $i \in \mathbf{C}$ be the standard square root of -1 and let $\bar{w} := \tau(w)$, where $w \in \mathbf{C}$. Under the transformation $x_{2j-1} := z_j + i z_{n+j}$ and $x_{2j} := z_j - i z_{n+j}$ (with $j \in \{1, \ldots, n\}$), we have

$$\mathfrak{Q}_n(x) = z_1^2 + z_2^2 + \cdots + z_{2n}^2 \text{ and } \mathfrak{H}_n^*(x) = 2i(-z_1 \bar{z}_{n+1} + z_{n+1} \bar{z}_1 - \cdots - z_n \bar{z}_{2n} + z_{2n} \bar{z}_n).$$

From the very definition of $\mathbf{SO}^*(2n)$ we get that (c) holds.

For $s \in \mathbf{N}$ we have standard closed embedding monomorphisms

$$\mathbf{GL}_{snS_2} \hookrightarrow \mathbf{Sp}_{2snS_2} \text{ and } \mathrm{Res}_{S_2/S_1} \mathbf{Sp}_{2snS_2} \hookrightarrow \mathbf{Sp}_{4snS_1}.$$

Therefore $\mathrm{Res}_{S_2/S_1} \mathcal{D}_{nS_2}$ is (via $\mathrm{Res}_{S_2/S_1} \rho_{nS_2}$) naturally a closed subgroup scheme of $\mathrm{Res}_{S_2/S_1} \mathbf{GL}_{2nS_2}$ and therefore also of $\mathrm{Res}_{S_2/S_1} \mathbf{Sp}_{4nS_2}$ and of $\mathbf{Sp}_{8nS_1}$. Thus (d) holds.$\square$

2.2.5. The case of number fields. Let F_0 be a totally imaginary quadratic extension of $\mathbf{Q}$ in which 2 splits. Let $S_1 := \mathbf{Z}_{(2)}$ and $S_2 := F_{0(2)}$; thus τ is the nontrivial $\mathbf{Z}_{(2)}$-automorphism of $F_{0(2)}$ and the reductive group scheme $\mathcal{D}_{n,\tau}$ is a form of $\mathcal{D}_n$. As 2 splits in F_0 and as $\mathcal{D}_{n,\tau}$ splits over $F_{0(2)}$ (cf. Lemma 2.2.4 (b)), $\mathcal{D}_{n,\tau}$ splits over $\mathbf{Z}_2$. Let F_1 be a totally real, finite Galois extension of $\mathbf{Q}$ unramified above 2 and of degree $q \in \mathbf{N}$. Let $F_2 := F_1 \otimes_{\mathbf{Q}} F_0$; it is a totally imaginary, Galois extension of $\mathbf{Q}$ that has degree $2q$ and that is unramified above 2. We view $N_1 := F_{1(2)}^{2n}$ as a free $\mathbf{Z}_{(2)}$-module of rank $2nq$ and $N_2 := F_{2(2)}^{2n}$ as a free $F_{0(2)}$-module of rank $2nq$.

See Subsection 2.1 and Subsubsection 2.2.3 for $\mathfrak{Q}_n$ and $\mathfrak{H}_n^*$. Let $\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}$ and $\mathfrak{H}_n^{*F_{1(2)}/\mathbf{Z}_{(2)}}$ be (the tensorizations with $F_{1(2)}$ over $\mathbf{Z}_{(2)}$ of) $\mathfrak{Q}_n$ and $\mathfrak{H}_n^*$ (respectively) but viewed as a quadratic form in $2nq$ variables on N_2 and as a skew hermitian quadratic form with respect to τ in $2nq$ variables on N_2 (respectively). If we fix an $F_{0(2)}$-linear isomorphism $c_2 : N_2 \tilde{\to} F_{0(2)}^{2nq}$, then for $w = (w_1, \ldots, w_{2nq}) \in N_2 \tilde{\to} F_{0(2)}^{2nq}$ we have

$$\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}(w) = \mathfrak{Q}_n(x) \text{ and } \mathfrak{H}_n^{*F_{1(2)}/\mathbf{Z}_{(2)}}(w) = \mathfrak{H}_n^*(x),$$

where $x = (x_1, \ldots, x_{2n}) := w \in F_{2(2)}^{2n} = N_2$ is computed via the standard $F_{2(2)}$-basis for $F_{2(2)}^{2n}$. By taking c_2 to be the natural tensorization of a $\mathbf{Z}_{(2)}$-linear isomorphism $c_1 : N_1 \tilde{\to} \mathbf{Z}_{(2)}^{2nq}$, we can also view naturally $\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}$ as a quadratic form in $2nq$ variables on the $\mathbf{Z}_{(2)}$-module N_1 and thus also on the $F_{0(2)}$-module $N_1 \otimes_{\mathbf{Z}_{(2)}} F_{0(2)}$.

Let $\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}$ be the identity component of the subgroup scheme of $\mathrm{Res}_{F_{0(2)}/\mathbf{Z}_{(2)}} \mathbf{GL}_{N_2}$ that fixes both $\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}$ and $\mathfrak{H}_n^{*F_{1(2)}/\mathbf{Z}_{(2)}}$; it is a group scheme over $\mathbf{Z}_{(2)}$.

2.2.6. Lemma. *The following four properties hold:*

(a) *the group scheme* ${\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}}_{F_{1(2)}}$ *is isomorphic to* $\mathcal{D}_{nq,\tau\,F_{1(2)}}$*;*

(b) *the group scheme* $\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}$ *is reductive for* $n \geq 1$ *and semisimple for* $n \geq 2$*;*

(c) *the group scheme* $\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}$ *splits over* $F_{0(2)}$ *and thus also over* $\mathbf{Z}_2$*;*

(d) *if* $(\tilde{L}_{(2)}, \tilde{\psi}')$ *is a symplectic space over* $\mathbf{Z}_{(2)}$ *of rank* $8nq$*, then there exist* $\mathbf{Z}_{(2)}$*-monomorphisms of reductive group schemes*

$$\operatorname{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}} \mathcal{D}_{n,\tau\,F_{1(2)}} \hookrightarrow \mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}} \hookrightarrow \mathbf{Sp}(\tilde{L}_{(2)}, \tilde{\psi}') \tag{1}$$

which are closed embeddings and for which the following three things hold:

(i) *the Lie monomorphism* $\operatorname{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}} \mathcal{D}_{n,\tau\,F_{1(2)}}(\mathbf{R}) \hookrightarrow \mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}(\mathbf{R})$ *is isomorphic to the Lie monomorphism* $e_{n,q}$ *of Subsubsection 2.2.1;*

(ii) *the extension of* $\operatorname{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}} \mathcal{D}_{n,\tau\,F_{1(2)}} \hookrightarrow \mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}$ *to* $F_{2(2)}$ *is isomorphic to the extension of the monomorphism* $d_{n,q}$ *(of Subsection 2.1) to* $F_{2(2)}$*;*

(iii) *the faithful representation* ${\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}}_{F_{0(2)}} \hookrightarrow \mathbf{GL}_{\tilde{L}_{(2)} \otimes_{\mathbf{Z}_{(2)}} F_{0(2)}}$ *is isomorphic to four copies of the representation* $\rho_{nq\,F_{0(2)}}$*.*

Proof: As F_1 and F_2 are Galois extensions of $\mathbf{Q}$ unramified above 2, we have natural identifications $F_{1(2)} \otimes_{\mathbf{Z}_{(2)}} F_{1(2)} = F_{1(2)}^q$ and $F_{2(2)} \otimes_{\mathbf{Z}_{(2)}} F_{1(2)} = F_{2(2)}^q$ of $\mathbf{Z}_{(2)}$-algebras. Thus for $x = (x_1, \ldots, x_{2n}) \in N_2 \otimes_{\mathbf{Z}_{(2)}} F_{1(2)} = (F_{2(2)} \otimes_{\mathbf{Z}_{(2)}} F_{1(2)})^{2n}$ we can speak about the transformation

$$x_{2j-1} = (w_{(2j-2)q+1}, w_{(2j-2)q+3}, \ldots, w_{(2j-2)q+2q-1}) \in F_{2(2)} \otimes_{\mathbf{Z}_{(2)}} F_{1(2)} = F_{2(2)}^q \quad \text{and}$$

$$x_{2j} = (w_{(2j-2)q+2}, w_{(2j-2)q+4}, \ldots, w_{(2j-2)q+2q}) \in F_{2(2)} \otimes_{\mathbf{Z}_{(2)}} F_{1(2)} = F_{2(2)}^q,$$

where $j \in \{1, \ldots, n\}$ and $w := (w_1, \ldots, w_{2nq}) \in F_{2(2)}^{2nq}$. Thus by considering the composite isomorphism $N_2 \otimes_{\mathbf{Z}_{(2)}} F_{1(2)} \tilde{\to} F_{0(2)}^{2nq} \otimes_{\mathbf{Z}_{(2)}} F_{1(2)} \tilde{\to} F_{1(2)} \otimes_{\mathbf{Z}_{(2)}} F_{0(2)}^{2nq} = F_{2(2)}^{2nq}$ whose inverse is defined naturally by this transformation, we have the following two identities

$$\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}(w) = \sum_{s=1}^{nq} w_{2s-1} w_{2s} \quad \text{and} \quad \mathfrak{H}_n^{*F_{1(2)}/\mathbf{Z}_{(2)}}(w) = \sum_{s=1}^{2nq} (-1)^{s+1} w_s (1_{F_{1(2)}} \otimes \tau)(w_s).$$

Thus we can redefine ${\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}}_{F_{1(2)}}$ as the subgroup scheme of $\operatorname{Res}_{F_{2(2)}/F_{1(2)}} \mathcal{D}_{nq\,F_{2(2)}}$ that fixes $\mathfrak{H}_{nq}^*$. From this (a) follows. Part (b) is implied by (a) and Lemma 2.2.4 (a).

To check (c), we first remark that we have an identification $F_{2(2)} \otimes_{\mathbf{Z}_{(2)}} F_{0(2)} = F_{2(2)} \oplus F_{2(2)}$ of $\mathbf{Z}_{(2)}$-algebras. Thus, similar to the proof of Lemma 2.2.4 (b) we get

that ${\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}}_{F_{0(2)}}$ is isomorphic to the identity component of the subgroup scheme of $\mathbf{GL}_{N_1\otimes_{\mathbf{Z}_{(2)}}F_{0(2)}}$ that fixes $\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}$ but viewed as a quadratic form in $2nq$ variables on the $F_{0(2)}$-module $N_1\otimes_{\mathbf{Z}_{(2)}}F_{0(2)}$. We choose an $F_{0(2)}$-basis $\{y_1,\ldots,y_{2nq}\}$ for $N_1\otimes_{\mathbf{Z}_{(2)}}F_{0(2)}$ such that $\{y_1,y_3,\ldots,y_{2nq-1}\}$ (resp. $\{y_2,y_4,\ldots,y_{2nq}\}$) are formed by elements of the first, third, ..., $2q-1$-th (resp. second, fourth, ..., $2q$-th) $F_{1(2)}\otimes_{\mathbf{Z}_{(2)}}F_{0(2)}$ copy of $N_1\otimes_{\mathbf{Z}_{(2)}}F_{0(2)}=(F_{1(2)}\otimes_{\mathbf{Z}_{(2)}}F_{0(2)})^{2n}$. If i_1, $i_2\in\{1,\ldots,2nq\}$ are congruent modulo 2, then we have

$$\mathfrak{B}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}(y_{i_1},y_{i_2}):=\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}(y_{i_1}+y_{i_2})-\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}(y_{i_1})-\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}(y_{i_2})=0.$$

This implies the existence of an $F_{0(2)}$-basis $\{y_1',y_2,\ldots,y_{2nq-1}',y_{2nq}\}$ for $N_1\otimes_{\mathbf{Z}_{(2)}}F_{0(2)}$ that has the following two properties:

(iv) the $F_{0(2)}$-spans of $\{y_1,y_3,\ldots,y_{2q-1}\}$ and $\{y_1',y_3',\ldots,y_{2q-1}'\}$ are equal;

(v) the matrix representation of the bilinear form $\mathfrak{B}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}$ defined by $\mathfrak{Q}_n^{F_{1(2)}/\mathbf{Z}_{(2)}}$ with respect to the $F_{0(2)}$-basis $\{y_1',y_2,\ldots,y_{2nq-1}',y_{2nq}\}$ for $N_1\otimes_{\mathbf{Z}_{(2)}}F_{0(2)}$, is $J(2nq)$.

From (v) we get that ${\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}}_{F_{0(2)}}$ is isomorphic to $\mathcal{D}_{nq\,F_{0(2)}}$ and thus it is a split group scheme. Thus (c) holds.

We have a canonical $\mathbf{Z}_{(2)}$-monomorphism $\mathrm{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}}\mathcal{D}_{n,\tau\,F_{1(2)}}\hookrightarrow\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}$; more precisely, $\mathrm{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}}\mathcal{D}_{n,\tau\,F_{1(2)}}$ is the closed subgroup scheme of $\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}$ whose group of $\mathbf{Z}_{(2)}$-valued points is the maximal subgroup of $\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}(\mathbf{Z}_{(2)})$ which is formed by $F_{2(2)}$-linear automorphisms of $N_2=F_{2(2)}^{2n}$. We take $\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}\hookrightarrow\mathbf{Sp}(\tilde{L}_{(2)},\tilde{\psi}')$ to be the composite of the following five $\mathbf{Z}_{(2)}$-monomorphisms

$$\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}\hookrightarrow\mathrm{Res}_{F_{0(2)}/\mathbf{Z}_{(2)}}(\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}})_{F_{0(2)}}$$

$$\tilde{\to}\mathrm{Res}_{F_{0(2)}/\mathbf{Z}_{(2)}}\mathcal{D}_{nq\,F_{0(2)}}\hookrightarrow\mathrm{Res}_{F_{0(2)}/\mathbf{Z}_{(2)}}\mathbf{Sp}_{4nq\,F_{0(2)}}\hookrightarrow\mathbf{Sp}_{8nq\,\mathbf{Z}_{(2)}}\tilde{\to}\mathbf{Sp}(\tilde{L}_{(2)},\tilde{\psi}')$$

(cf. (c) for the second one and cf. proof of Lemma 2.2.4 (d) for the third and fourth ones).

Part (i) of (d) follows from the proof of (a): the variables $z_1^{(1)}$, ..., $z_{2n}^{(q)}$ we used to define the Lie monomorphism $e_{n,q}$ are the variables w_1, ..., w_{2nq} up to a natural permutation. The extension of $\mathrm{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}}\mathcal{D}_{n,\tau\,F_{1(2)}}$ to $F_{2(2)}$ is isomorphic to $\mathrm{Res}_{F_{2(2)}^q/F_{2(2)}}\mathcal{D}_{n,\tau\,F_{2(2)}}\tilde{\to}\mathcal{D}_{n,\tau\,F_{2(2)}}^{\;q}$ and therefore also to $\mathcal{D}_{n\,F_{2(2)}}^{\;q}$, cf. Lemma 2.2.4 (b). Moreover, the extension of $\mathcal{D}_{nq,\tau}^{F_{1(2)}/\mathbf{Z}_{(2)}}$ to $F_{2(2)}$ is isomorphic to $\mathcal{D}_{nq\,F_{2(2)}}$ (cf. (a) and (c)). Thus part (ii) of (d) follows easily from constructions. Part (iii) follows from (c) and Lemma 2.2.4 (d), cf. constructions. $\square$

3. Proof of the Basic Theorem

In this Section the following list of notations

$(\mathbf{L}_1)$ $\mathbf{S}$, $f:(G,\mathcal{X})\hookrightarrow(\mathbf{GSp}(W,\psi),\mathcal{S})$, $E(G,\mathcal{X}), v, O_{(v)}$, L, $L_{(2)}$, $G_{\mathbf{Z}_{(2)}}$, K_2, H_2, $\mathcal{B}$, $\mathcal{I}$

is as in Section 1. In order to prove the Basic Theorem (see Subsections 3.3 to 3.5), we will need few extra notations and properties (see Lemma 3.1 and Subsection 3.2). Let n, $r\in\mathbf{N}$ with $n\geq 2$ be such that the group $G^{\mathrm{der}}_{\mathbf{C}}$ is isomorphic to $\mathbf{SO}^r_{2n\mathbf{C}}$, cf. assumption 1.1 (v). Let $d:=\frac{rn(n-1)}{2}\in\mathbf{N}$. Let $B(\mathbf{F})$ be the field of fractions of $W(\mathbf{F})$.

Let $\mathcal{B}_1$ be the centralizer of $\mathcal{B}$ in $\mathrm{End}(L_{(2)})$. Let $G_{2\mathbf{Z}_{(2)}}$ be the centralizer of $\mathcal{B}$ in $\mathbf{GL}_{L_{(2)}}$; thus $G_{2\mathbf{Z}_{(2)}}$ is the reductive group scheme over $\mathbf{Z}_{(2)}$ of invertible elements of $\mathcal{B}_1$ and $\mathrm{Lie}(G_{2\mathbf{Z}_{(2)}})$ is the Lie algebra associated to $\mathcal{B}_1$. Due to the axiom 1.1 (i), the $W(\mathbf{F})$-algebra $\mathcal{B}_1\otimes_{\mathbf{Z}_{(2)}}W(\mathbf{F})$ is also a product of matrix $W(\mathbf{F})$-algebras. This implies that $G_{2W(\mathbf{F})}$ is a product of $\mathbf{GL}_{2n}$ groups schemes over $W(\mathbf{F})$. For basic terminology on involutions of semisimple Lie algebras we refer to [19, Subsection 3.3]. All simple factors of $(\mathcal{B}_j,\mathcal{I})\otimes_{\mathbf{Z}_2}W(\mathbf{F})$ have the same type which (due to the assumption 1.1 (v)) is the first orthogonal type. Thus the $W(\mathbf{F})$-monomorphism $G^{\mathrm{der}}_{W(\mathbf{F})}\hookrightarrow G_{2W(\mathbf{F})}$ between reductive group schemes is isomorphic to the product of r copies of $\rho_{nW(\mathbf{F})}$ and thus $G_{2W(\mathbf{F})}$ is isomorphic to $\mathbf{GL}^r_{2nW(\mathbf{F})}$.

3.1. Lemma. *There exists a totally real number field F of degree r and a semisimple group G_{oF} over F such that the following two properties hold:*

- **(i)** *the derived group G^{der} is isomorphic to $\mathrm{Res}_{F/\mathbf{Q}}G_{oF}$;*
- **(ii)** *the group G_{oF} is a form of $\mathbf{SO}^+_{2nF}$.*

Proof: Let F be the center of $\mathcal{B}_1[\frac{1}{2}]$. Due to the axiom 1.1 (ii), the $\mathbf{Q}$–algebra $\mathcal{B}_1[\frac{1}{2}]$ is simple. Thus F is a field. As $G_{2W(\mathbf{F})}$ is isomorphic to $\mathbf{GL}^r_{2nW(\mathbf{F})}$, we have $[F:\mathbf{Q}]=r$. Let G_3 be the reductive group over F of invertible elements of (warning!) the F-algebra $\mathcal{B}_1[\frac{1}{2}]$. We can identify G_2 with $\mathrm{Res}_{F/\mathbf{Q}}G_3$. As G^{der} is a subgroup of G_2 such that $G^{\mathrm{der}}_{\mathbf{C}}$ is a product of subgroups of the factors of the following product $G_{2\mathbf{C}}=(\mathrm{Res}_{F/\mathbf{Q}}G_3)_{\mathbf{C}}=\prod_{j:F\hookrightarrow\mathbf{C}}G_3\times_{F\,j}\mathbf{C}$, there exists a subgroup G_{oF} of G_3 such that we can identify G^{der} with $\mathrm{Res}_{F/\mathbf{Q}}G_{oF}$. More precisely, we have a natural product decomposition $F\otimes_{\mathbf{Q}}F=F\times F^{\perp}$ of étale F-algebras and therefore we can identify G_{2F} with $G_3\times_F\mathrm{Res}_{F^{\perp}/F}G_{3F^{\perp}}$; we can now take $G_{oF}:=G_3\cap G^{\mathrm{der}}_F$, the intersection being taken inside G_{2F}. Thus (i) holds.

It is easy to see that each group $G_{oF}\times_{F\,j}\mathbf{C}$ is isomorphic to $\mathbf{SO}_{2n\mathbf{C}}$. Thus (ii) also holds. As each simple factor of $G^{\mathrm{ad}}_{\mathbf{R}}\tilde{\to}\mathrm{Res}_{F\otimes_{\mathbf{Q}}\mathbf{R}/\mathbf{R}}G_{oF}\times_F F\otimes_{\mathbf{Q}}\mathbf{R}$ is absolutely simple (cf. [4, Subsubsection 2.3.4 (a)]), the $\mathbf{R}$-algebra $F\otimes_{\mathbf{Q}}\mathbf{R}$ is isomorphic to $\mathbf{R}^r$. Thus the number field F is indeed totally real. $\square$

3.2. On $G^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$. Let κ be the set of primes of F that divide 2. We have a natural product decomposition

$$F\otimes_{\mathbf{Q}}\mathbf{Q}_2=\prod_{j\in\kappa}F_j$$

into 2-adic fields. Due to Lemma 3.1 (i), we can identify $G^{\mathrm{der}}_{\mathbf{Q_2}}$ with $\prod_{j\in\kappa}\mathrm{Res}_{F_j/\mathbf{Q_2}}G_{oF_j}$. Thus we can also identify $G^{\mathrm{der}}_{B(\mathbf{F})}$ with $\prod_{j\in\kappa}\mathrm{Res}_{F_j\otimes_{\mathbf{Q_2}}B(\mathbf{F})/B(\mathbf{F})}G_{oF_j}\times_{F_j}B(\mathbf{F})$. As $G_{\mathbf{Z_{(2)}}}$ splits over $W(\mathbf{F})$, the group $G^{\mathrm{der}}_{B(\mathbf{F})}$ is split. Thus the $B(\mathbf{F})$-algebra $F_j\times_{\mathbf{Q_2}}B(\mathbf{F})$ is isomorphic to a product of copies of $B(\mathbf{F})$. Thus each field F_j is unramified over $\mathbf{Q}_2$ i.e., F is unramified above 2. Thus the finite $\mathbf{Z_{(2)}}$-algebra $F_{(2)}$ (see Section 2) is étale.

As $\mathcal{B}_1$ is a semisimple $\mathbf{Z}_{(2)}$-algebra, it is also a semisimple $F_{(2)}$-algebra. Let $G_{3F_{(2)}}$ be the reductive group scheme over $F_{(2)}$ of invertible elements of the $F_{(2)}$-algebra $\mathcal{B}_1$. We can identify $G_{2\mathbf{Z}_{(2)}}$ with $\mathrm{Res}_{F_{(2)}/\mathbf{Z}_{(2)}}G_{3F_{(2)}}$. Let $G_{oF_{(2)}}$ be the Zariski closure of G_{oF} in $G_{3F_{(2)}}$. We can identify $G^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$ with $\mathrm{Res}_{F_{(2)}/\mathbf{Z}_{(2)}}G_{oF_{(2)}}$ and this implies that $G_{oF_{(2)}}$ is a semisimple group scheme over $F_{(2)}$.

Let O_j be the ring of integers of F_j. For a later use we point out that we can identify $G^{\mathrm{der}}_{\mathbf{Z}_2}$ with $\prod_{j\in\kappa}\mathrm{Res}_{O_j/\mathbf{Z}_2}G_{oO_j}$. Thus we can also identify κ with the set of factors of $G^{\mathrm{der}}_{\mathbf{Z}_2}$ that are Weil restrictions of semisimple $\mathbf{SO}_{2n}$ group schemes.

3.3. A twisting process. Let $\mathcal{D}_{n,\tau}$ be the form of $\mathcal{D}_n$ introduced in Subsubsection 2.2.5. The Lie group $G^{\mathrm{der}}_{\mathbf{R}}(\mathbf{R})$ is isomorphic to $\mathbf{SO}^*(2n)^r$, cf. [14, Subsections 2.6 and 2.7]. This property implies that each connected component of $\mathcal{X}$ is a product of r copies of the irreducible hermitian symmetric domain associated to $\mathbf{SO}^*(2n)$ and thus (cf. [9, Ch. X, §6, Table V]) we have

$$\dim_{\mathbf{C}}(\mathcal{X})=d=\frac{rn(n-1)}{2}.$$

The mentioned property also implies that the semisimple group schemes $G_{oF_{(2)}}$ and $\mathcal{D}_{n,\tau\,F_{(2)}}$ become isomorphic under extensions via $\mathbf{Z}_{(2)}$-monomorphisms $F_{(2)}\hookrightarrow\mathbf{R}$, cf. also Lemma 2.2.6 (i).

We have short exact sequences $0\to G^{\mathrm{ad}}_{oF_{(2)}}\to\mathrm{Aut}(G^{\mathrm{ad}}_{oF_{(2)}})\to\boldsymbol{\mu}_{2\,F}\to 0$ (see [6, Vol. III, Exp. XXIV, Thm. 1.3]) and $0\to\boldsymbol{\mu}_{2\,F_{(2)}}\to G_{oF_{(2)}}\to G^{\mathrm{ad}}_{oF_{(2)}}\to 0$. The nontrivial torsors of $\boldsymbol{\mu}_{2\,F}\tilde{\to}(\mathbf{Z}/2\mathbf{Z})_F$ correspond to quadratic field extensions of F. We easily get that there exists a smallest totally real field extension F_{in} of F of degree at most 2 and such that $G_{oF_{\mathrm{in}}}$ and $\mathcal{D}_{n,\tau\,F_{\mathrm{in}}}$ are inner forms of each other. The extensions of $G_{oF_{(2)}}$ and $\mathcal{D}_{n,\tau\,F_{(2)}}$ via $\mathbf{Z}_{(2)}$-monomorphisms $F_{(2)}\hookrightarrow W(\mathbf{F})$ are isomorphic to $\mathbf{SO}^+_{2n\,W(\mathbf{F})}$. Thus the field F_{in} is unramified above all primes of F that divide 2.

Let $\gamma_1\in H^1(F_{\mathrm{in}},G^{\mathrm{ad}}_{oF_{\mathrm{in}}})$ be the class that defines the inner form $\mathcal{D}_{n,\tau\,F_{\mathrm{in}}}$ of $G_{oF_{\mathrm{in}}}$. Let $\gamma_2\in H^2(F_{\mathrm{in}},\boldsymbol{\mu}_{2\,F_{\mathrm{in}}})$ be the image of γ_1. Let $M(\gamma_2)$ be the central semisimple F_{in}-algebra that defines γ_2; it is either F_{in} itself or a nontrivial form of $M_2(F_{\mathrm{in}})$. We know that γ_2 becomes trivial under each embedding of F_{in} into either $\mathbf{R}$ or $B(\mathbf{F})$. Thus from [8, Lemma 5.5.3] we get that there exists a maximal torus of the group scheme of invertible elements of $M(\gamma_2)$ that is defined by a totally real number field extension F_{oin} of F_{in} of degree at most 2 and unramified above the primes of F_{in} that divide 2. Let F_1 be the Galois extension of $\mathbf{Q}$ generated by F_{oin}; it is totally real and unramified above 2. The

image of γ_2 in $H^2(F_1, \boldsymbol{\mu}_{2F_1})$ is the trivial class and thus we can speak about the class

$$\gamma_0 \in H^1(F_{1(2)}, \mathcal{D}_{n,\tau F_{1(2)}}) = H^1(\mathbf{Z}_{(2)}, \mathrm{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}} \mathcal{D}_{n,\tau F_{1(2)}})$$

that defines the inner twist $G_{oF_{1(2)}}$ of $\mathcal{D}_{n,\tau F_{1(2)}}$.

Let $q := [F_1 : \mathbf{Q}]$; we have $q \in r\mathbf{N}$ and our notations match with the ones of Subsubsection 2.2.5. Let $\tilde{\gamma}_0$ be the image of γ_0 in $H^1(\mathbf{Z}_{(2)}, \mathbf{Sp}(\tilde{L}_{(2)}, \tilde{\psi}'))$ via the $\mathbf{Z}_{(2)}$-monomorphisms of (1). We define $\tilde{W} := \tilde{L}_{(2)}[\frac{1}{2}]$. The image of $\tilde{\gamma}_0$ in $H^1(\mathbf{Q}, \mathbf{Sp}(\tilde{W}, \tilde{\psi}'))$ is trivial. [Argument: based on [8, Main Thm.], it suffices to show that the image of γ_0 in $H^1(\mathbf{R}, \mathbf{Sp}(\tilde{W} \otimes_{\mathbf{Q}} \mathbf{R}, \tilde{\psi}'))$ is trivial; but this is so as the image of γ_0 in $H^1(\mathbf{R}, (\mathrm{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}} \mathcal{D}_{n,\tau F_{1(2)}})_{\mathbf{R}})$ is trivial.] This implies that $\tilde{\gamma}_0$ is the trivial class. Thus by twisting the $\mathbf{Z}_{(2)}$-monomorphisms of (1) via γ_0, we get $\mathbf{Z}_{(2)}$-monomorphisms of the form

$$\mathrm{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}} G_{oF_{1(2)}} \hookrightarrow G'^{\mathrm{der}}_{\mathbf{Z}_{(2)}} \hookrightarrow \mathbf{Sp}(\tilde{L}_{(2)}, \tilde{\psi}'),$$

where $G'^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$ is a form of $\mathcal{D}_{nq}$. Using the natural $\mathbf{Z}_{(2)}$-monomorphism

$$G^{\mathrm{der}}_{\mathbf{Z}_{(2)}} = \mathrm{Res}_{F_{(2)}/\mathbf{Z}_{(2)}} G_{oF_{(2)}} \hookrightarrow \mathrm{Res}_{F_{1(2)}/\mathbf{Z}_{(2)}} G_{oF_{1(2)}},$$

we end up with a sequence of closed embedding $\mathbf{Z}_{(2)}$-monomorphisms

$$G^{\mathrm{der}}_{\mathbf{Z}_{(2)}} \hookrightarrow G'^{\mathrm{der}}_{\mathbf{Z}_{(2)}} \hookrightarrow \mathbf{Sp}(\tilde{L}_{(2)}, \tilde{\psi}'). \tag{2}$$

As in [19, Subsection 3.5] we argue that the normal subgroup $G^0 := G \cap \mathbf{Sp}(W, \psi)$ of G is connected and therefore reductive. Let $G^0_{\mathbf{Z}_{(2)}}$ be the Zariski closure of G^0 in $G_{\mathbf{Z}_{(2)}}$. As in [19, Subsubsection 3.5.1] we argue that we have $G^0_{\mathbf{Z}_{(2)}} = G^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$. Thus $G_{\mathbf{Z}_{(2)}}$ is the flat, closed subgroup scheme of $\mathbf{GL}_{\tilde{L}_{(2)}}$ generated by $G^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$ and by the center of $\mathbf{GL}_{\tilde{L}_{(2)}}$. Let $G'_{\mathbf{Z}_{(2)}}$ be the flat, closed subgroup scheme of $\mathbf{GL}_{\tilde{L}_{(2)}}$ generated by $G'^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$ and by the center of $\mathbf{GL}_{\tilde{L}_{(2)}}$; it is a form of $\mathbf{GSO}^+_{2nq\mathbf{Z}_{(2)}}$ and thus a reductive group scheme. As the group $\mathcal{D}^{F_{1(2)}/\mathbf{Z}_{(2)}}_{nq,\tau}$ of Subsubsection 2.2.5 splits over $\mathbf{Z}_2$ (cf. Lemma 2.2.6 (c)) and as the class γ_0 has a trivial image in $H^1(\mathbf{Z}_2, \mathrm{Res}_{F_{1(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2/\mathbf{Z}_2} \mathcal{D}_{n,\tau F_{1(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2})$ (cf. Lang's theorem and the fact that the ring $\mathbf{Z}_2$ is henselian), the group scheme $G'^{\mathrm{der}}_{\mathbf{Z}_2}$ is split. Thus the extension of $G'_{\mathbf{Z}_{(2)}}$ to $\mathbf{Z}_2$ splits and therefore it is isomorphic to $\mathbf{GSO}^+_{2nq\mathbf{Z}_2}$. Thus the property 1.2 (i) holds.

3.4. The new Shimura pair $(G', \mathcal{X}')$. We define $G' := G'_{\mathbf{Z}_{(2)}} \times_{\mathbf{Z}_{(2)}} \mathbf{Q}$. Let $\mathcal{X}'$ be the $G'(\mathbf{R})$-conjugacy class of the composite of any element $h : \mathbf{S} \hookrightarrow G_{\mathbf{R}}$ of $\mathcal{X}$ with the $\mathbf{R}$-monomorphism $G_{\mathbf{R}} \hookrightarrow G'_{\mathbf{R}}$. The Lie monomorphism $G^{\mathrm{der}}_{\mathbf{R}}(\mathbf{R}) \hookrightarrow G'^{\mathrm{der}}_{\mathbf{R}}(\mathbf{R})$ can be identified with the composite of a diagonal Lie monomorphism $\mathbf{SO}^*(2n)^r \hookrightarrow \mathbf{SO}^*(2n)^q$ with the Lie monomorphism $e_{n,q} : \mathbf{SO}^*(2n)^q \hookrightarrow \mathbf{SO}^*(2nq)$ (cf. the constructions of

Subsection 3.3 and Lemma 2.2.6 (d)). As $Ad \circ h(i)$ is a Cartan involution of Lie($G^{\mathrm{ad}}_{\mathbf{R}}$) (cf. beginning of Section 1), the image S_h through h of the $\mathbf{SO}(2) = \mathbf{SO}^*(2)$ Lie subgroup of $\mathbf{S}(\mathbf{R})$, is the center of a maximal compact Lie subgroup of $\mathbf{SO}^*(2n)^r = G^{\mathrm{der}}(\mathbf{R})$. But all maximal compact Lie subgroups of $\mathbf{SO}^*(2n)^r$ are $\mathbf{SO}^*(2n)^r$-conjugate (see [9, Ch. VI, §2]). By combining the last two sentences with Lemma 2.2.2 (c), we get that the Lie subgroup S_h of $\mathbf{SO}^*(2nq) = G'^{\mathrm{der}}(\mathbf{R})$ is $\mathbf{SO}^*(2nq)$-conjugate to $C_{1,nq} = \mathrm{Im}(s_{1,nq})$. Thus the centralizer of S_h in $\mathbf{SO}^*(2nq) = G'^{\mathrm{der}}(\mathbf{R})$ is a maximal compact Lie subgroup of $\mathbf{SO}^*(2nq)$ that is isomorphic to $\mathbf{U}(nq)$ (cf. Lemma 2.2.2 (c)). This implies that the inner conjugation through $h(i)$ is a Cartan involution of Lie($G'^{\mathrm{der}}_{\mathbf{R}}$) = Lie($G'^{\mathrm{ad}}_{\mathbf{R}}$), cf. the classification of Cartan involutions of [9, Ch. X, §2].

The representation of $G^{\mathrm{der}}_{\mathbf{C}}$ on $\tilde{W} \otimes_{\mathbf{Q}} \mathbf{C}$ is a direct sum of standard representations of dimension $2n$ of the $\mathbf{SO}_{2n\mathbf{C}}$ factors of $G^{\mathrm{der}}_{\mathbf{C}}$. Thus the Hodge $\mathbf{Q}$–structure on $\tilde{W}$ defined by h has the same type as the Hodge $\mathbf{Q}$–structure on W defined by h and thus it is of type $\{(-1,0),(0,-1)\}$. As the subgroup $\mathbf{SO}^*(2n)^r = G^{\mathrm{der}}(\mathbf{R})$ of $G'^{\mathrm{der}}(\mathbf{R})$ is not compact, the group G'^{ad} has no simple factor which over $\mathbf{R}$ is compact. Based on the last two sentences and on the last sentence of the previous paragraph, we get that Deligne's axioms of the first paragraph of Section 1 hold for the pair $(G',\mathcal{X}')$. Thus $(G',\mathcal{X}')$ is a Shimura pair.

3.5. End of the proof of the Basic Theorem. Let $\tilde{\mathfrak{A}}$ be the free $\mathbf{Z}_{(2)}$-module of alternating forms on $\tilde{L}_{(2)}$ fixed by $G'^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$. There exist elements of $\tilde{\mathfrak{A}} \otimes_{\mathbf{Z}_{(2)}} \mathbf{R}$ that define polarizations of the Hodge $\mathbf{Q}$–structure on $\tilde{W}$ defined by $h \in \mathcal{X}$, cf. [4, Cor. 2.3.3]. Thus the real vector space $\tilde{\mathfrak{A}} \otimes_{\mathbf{Z}_{(2)}} \mathbf{R}$ has a non-empty, open subset of such polarizations (cf. [4, Subsubsection 1.1.18 (a)]). A standard application to $\tilde{\mathfrak{A}}$ of the approximation theory for independent valuations, implies the existence of an alternating form $\tilde{\psi}$ on $\tilde{L}_{(2)}$ that is fixed by $G'^{\mathrm{der}}_{\mathbf{Z}_{(2)}}$, that is congruent modulo $2\mathbf{Z}_{(2)}$ to $\tilde{\psi}'$, and that defines a polarization of the Hodge $\mathbf{Q}$–structure on $\tilde{W}$ defined by $h \in \mathcal{X}$. As $\tilde{\psi}$ is congruent modulo $2\mathbf{Z}_{(2)}$ to $\tilde{\psi}'$, it is a perfect, alternating form on $\tilde{L}_{(2)}$.

We get injective maps $\tilde{f} : (G,\mathcal{X}) \hookrightarrow (\mathbf{GSp}(\tilde{W},\tilde{\psi}),\tilde{\mathcal{S}})$ and $\tilde{f}' : (G',\mathcal{X}') \hookrightarrow (\mathbf{GSp}(\tilde{W},\tilde{\psi}),\tilde{\mathcal{S}})$ of Shimura pairs. Let $\tilde{L}$ be a $\mathbf{Z}$-lattice of $\tilde{W}$ such that $\tilde{\psi}$ induces a perfect alternating form on it and we have $\tilde{L}_{(2)} = \tilde{L} \otimes_{\mathbf{Z}} \mathbf{Z}_{(2)}$. Let $\tilde{\mathcal{B}}$, $\tilde{\mathcal{B}}'$, and v' be as in the property 1.2 (ii). Let $\tilde{\mathcal{I}}$ be the involution of $\mathrm{End}(\tilde{L}_{(2)})$ defined by $\tilde{\psi}$. We check that the axioms 1.1 (i) to (v) hold for the quadruple $(\tilde{f},\tilde{L},v,\tilde{\mathcal{B}})$. Obviously the axiom 1.1 (v) holds. We know that the axiom 1.1 (iv) holds for $(\tilde{f},\tilde{L},v,\tilde{\mathcal{B}})$, cf. the last paragraph of Subsection 3.3. From Lemma 2.2.6 (ii) we get that the representation of $G^{\mathrm{der}}_{W(\mathbf{F})} = G^0_{W(\mathbf{F})}$ on $\tilde{L}_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F})$ is isomorphic to the direct sum of a finite number of copies of the representation $\rho_{nW(\mathbf{F})}$. As $n \geq 2$, the fibres of $\rho_{nW(\mathbf{F})}$ are absolutely simple representations. From the last two sentences we get that $\tilde{\mathcal{B}} \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F})$ is a product of matrix $W(\mathbf{F})$-algebras. Thus the axiom 1.1 (i) holds for $(\tilde{f},\tilde{L},v,\tilde{\mathcal{B}})$.

As $G^{\mathrm{der}}_{F_1}$ is a product of groups that are forms of $\mathbf{SO}^+_{2nF_1}$ permuted transitively by $\mathrm{Gal}(F_1/\mathbf{Q})$, the $\mathbf{Q}$–algebra $\tilde{\mathcal{B}}[\frac{1}{2}]$ is simple. Thus the axiom 1.1 (ii) holds for $(\tilde{f},\tilde{L},v)$.

The fact that the axiom 1.1 (iii) holds for $(\tilde{f}, \tilde{L}, v)$ is a standard consequence of the fact that G is generated by the center of $\mathbf{GL}_{\tilde{W}}$ and by G^{der} and of the description of the representation of $G^{\text{der}}_{W(\mathbf{F})}$ on $\tilde{L}_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F})$. We conclude that the quadruple $(\tilde{f}, \tilde{L}, v, \tilde{\mathcal{B}})$ is a hermitian orthogonal standard PEL situation in mixed characteristic $(0,2)$. Similarly we argue that $(\tilde{f}', \tilde{L}, v', \tilde{\mathcal{B}}')$ is a hermitian orthogonal standard Hodge situation in mixed characteristic $(0,2)$. Thus the property 1.2 (ii) holds. From the construction of (2) and Lemma 2.2.6 (ii) and (iii) we get:

(i) the natural $W(\mathbf{F})$-monomorphism $G^{\text{der}}_{W(\mathbf{F})} \hookrightarrow G'^{\text{der}}_{W(\mathbf{F})}$ is the composite of a diagonal $W(\mathbf{F})$-monomorphism $\mathcal{D}^r_{n W(\mathbf{F})} \hookrightarrow \mathcal{D}^q_{n W(\mathbf{F})}$ with a standard $W(\mathbf{F})$-monomorphism $d_{n,q W(\mathbf{F})} : \mathcal{D}^q_{n W(\mathbf{F})} \hookrightarrow \mathcal{D}_{nq W(\mathbf{F})}$;

(ii) the faithful representation $G'^{\text{der}}_{W(\mathbf{F})} \hookrightarrow \mathbf{GL}_{\tilde{L}_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F})}$ is isomorphic to the direct sum of four copies of the representation $\rho_{nq W(\mathbf{F})}$.

From (i), (ii), and Subsubsection 2.2.1 we get that $G^{\text{der}}_{W(\mathbf{F})}$ is the identity component of the subgroup scheme of $G'^{\text{der}}_{W(\mathbf{F})}$ that centralizes $\tilde{\mathcal{B}} \otimes_{\mathbf{Z}_{(2)}} W(\mathbf{F})$. This implies that the property 1.2 (iii) also holds. This ends the proof of the Basic Theorem. $\square$

4. Basic crystalline properties

Until the end we use the notations of the list $(\mathbf{L}_1)$ of Section 3 and of the following new list of notations

$(\mathbf{L}_2)$ $\qquad n,\ r,\ d,\ \mathcal{B}_1,\ G_2,\ G_{2\mathbf{Z}_{(2)}},\ B(\mathbf{F}),\ F,\ G_{oF},\ \kappa,\ j \in \kappa,\ F_j,\ F_{(2)},\ G_{oF_{(2)}}, O_j$

introduced in Section 3. Let $(\mathcal{A}, \Lambda)$ be the pull back to $\mathcal{N}$ of the universal principally polarized abelian scheme over $\mathcal{M}$. Let k be an arbitrary algebraically closed field of characteristic 2 and of countable transcendental degree. We fix a $\mathbf{Z}_{(2)}$-embedding $O_{(v)} \hookrightarrow W(k)$ into the ring of Witt vectors with coefficients in k. All pulls back to either $W(k)$ or k of $O_{(v)}$-schemes, will be with respect to this $\mathbf{Z}_{(2)}$-embedding. For a $W(k)$-morphism $y : \text{Spec}(k) \to \mathcal{N}_{W(k)}$ let

$$(A, \lambda_A) := y^*((\mathcal{A}, \Lambda) \times_{\mathcal{N}} \mathcal{N}_{W(k)}).$$

Let (M, Φ, ψ_M) be the principally quasi-polarized F-crystal over k of (A, λ_A). The pair (M, Φ) is a Dieudonné module over k of rank $\dim_{\mathbf{Q}}(W)$ and ψ_M is a perfect, alternating form on M. We denote also by ψ_M the perfect, alternating form on M^* induced naturally by ψ_M.

For $b \in \mathcal{B}$, we denote also by b the $\mathbf{Z}_{(2)}$-endomorphism of $\mathcal{A}$ defined naturally by b. We denote also by b different de Rham (crystalline) realizations of $\mathbf{Z}_{(2)}$-endomorphisms that correspond to b. Thus we will speak about the $\mathbf{Z}_{(2)}$-monomorphism $\mathcal{B}^{\text{opp}} \hookrightarrow \text{End}(M)$ that makes M to be a $\mathcal{B}^{\text{opp}} \otimes_{\mathbf{Z}_{(2)}} W(k)$-module and M^* to be a $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$-module;

here $\mathcal{B}^{\mathrm{opp}}$ is the opposite $\mathbf{Z}_{(2)}$-algebra of $\mathcal{B}$. In this Section we recall basic crystalline properties that are (or are proved) as in [19].

4.1. Extra tensors. Let $(v_\alpha)_{\alpha\in\mathcal{J}}$ be a family of tensors of $\mathcal{T}(W)$ such that G is the subgroup of $\mathbf{GL}_W$ that fixes v_α for all $\alpha\in\mathcal{J}$, cf. [5, Prop. 3.1 (c)]. We choose the set $\mathcal{J}$ such that $\mathcal{B}\subseteq\mathcal{J}$ and for $b\in\mathcal{B}$ we have $v_b=b\in\mathrm{End}(W)=W\otimes_{\mathbf{Q}}W^*$.

Let $(\mathcal{B}_1\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2,\mathcal{I})=\oplus_{j\in\kappa}(\mathcal{B}_j,\mathcal{I})$ be the product decomposition of $(\mathcal{B}_1\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2,\mathcal{I})$ into simple factors. Each $\mathcal{B}_j$ is a two sided ideal of $\mathcal{B}_1\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$ that is a simple $\mathbf{Z}_2$-algebra and whose center is the ring of integers O_j of the 2-adic field F_j.

As F_j is unramified over $\mathbf{Q}_2$ (see Subsection 3.2), O_j is a finite, étale $\mathbf{Z}_2$-algebra. We can identify $\mathcal{B}_j\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$ with $\mathrm{End}(\mathcal{V}_j)$, where $\mathcal{V}_j$ is a free O_j-module of rank $2n$. Let $s_j\in\mathbf{N}\setminus\{1\}$ be such that as $\mathcal{B}_1\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$-modules we can identify

$$L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2=\oplus_{j\in\kappa}\mathcal{V}_j^{s_j}. \tag{3}$$

[We have $s_j\neq 1$, as the representation of $G^{\mathrm{der}}_{\mathbf{Z}_2}=G^0_{\mathbf{Z}_2}$ on $L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$ is symplectic.] We can redefine the direct summand $\mathcal{V}_j^{s_j}$ of $L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$ as the maximal $\mathbf{Z}_2$-submodule that is generated by non-trivial, simple $\mathrm{Res}_{O_j/\mathbf{Z}_2}G_{oF_{(2)}}\times_{F_{(2)}}O_j$-submodules (we recall that $\mathrm{Res}_{O_j/\mathbf{Z}_2}G_{oF_{(2)}}\times_{F_{(2)}}O_j$ is a direct factor of $G^{\mathrm{der}}_{\mathbf{Z}_2}$ introduced in Subsection 3.2).

Let b_j be a perfect bilinear form on the O_j-module $\mathcal{V}_j$ that defines the involution $\mathcal{I}$ of $\mathcal{B}_1\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$, cf. [19, Lemma 3.3.1 (a)]. Thus b_j is unique up to a $\mathbf{G}_m(O_j)$-multiple (cf. [19, Lemma 3.3.1 (b)]), it is fixed by $G^{\mathrm{der}}_{\mathbf{Z}_2}=G^0_{\mathbf{Z}_2}$, and it is symmetric (as $(\mathcal{B}_j,\mathcal{I})$ is of orthogonal first type). Let $b_0:=\oplus_{j\in\kappa}b_j^{s_j}$; it is a perfect, symmetric bilinear form on the $\mathbf{Z}_2$-module $L_{(2)}\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2$ that is fixed by $G^{\mathrm{der}}_{\mathbf{Z}_2}=G^0_{\mathbf{Z}_2}$. Thus the $\mathbf{Z}_2$-span of b is normalized by $G_{\mathbf{Z}_2}$.

4.2. Lifts. As in [19, Subsection 4.1] we argue that there exists a compact, open subgroup H_0 of $G(\mathbf{A}_f^{(2)})$ such that $\mathcal{N}$ is a pro-étale cover of the quasi-projective, normal $O_{(v)}$-scheme $\mathcal{N}/H_0$. The flat, finite type morphism $\mathcal{N}_{W(k)}/H_0\to\mathrm{Spec}(W(k))$ has quasi-sections whose images contain the k-valued point of $\mathcal{N}_{W(k)}/H_0$ defined by y (cf. [7, Cor. (17.16.2)]). This implies that the $W(k)$-morphism $y:\mathrm{Spec}(k)\to\mathcal{N}_{W(k)}$ has a lift

$$z:\mathrm{Spec}(V)\to\mathcal{N}_{W(k)},$$

where V is a finite, discrete valuation ring extension of $W(k)$. We define $(A_V,\lambda_{A_V}):=z^*((\mathcal{A},\Lambda)_{\mathcal{N}_{W(k)}})$. Let $B(k):=W(k)[\frac{1}{2}]$. As in [19, Subsection 4.2] we argue that:

(a) for each $\alpha\in\mathcal{J}$ there exists a tensor $t_\alpha\in\mathcal{T}(M[\frac{1}{2}])$ that correspond naturally to v_α via Fontaine comparison theory for A_V;

(b) there exists a $B(k)$-linear isomorphism $j_y:L_{(2)}\otimes_{\mathbf{Z}_{(2)}}B(k)\tilde{\to}M^*[\frac{1}{2}]$ that takes ψ to ψ_M and takes v_α to t_α for all $\alpha\in\mathcal{J}$.

Let J be the Zariski closure in $\mathbf{GL}_M$ of the subgroup $J_{B(k)}$ of $\mathbf{GL}_{M[\frac{1}{2}]}$ that fixes t_α for all $\alpha\in\mathcal{J}$. The existence of j_y implies that $J_{B(k)}$ is isomorphic to $G_{B(k)}$ and thus it

is a reductive group. Let $j \in \kappa$. Each projection of $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$ on a factor $\mathcal{V}_j$ of (3) along the direct sum of the other factors of (3), is an element of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$. Thus to (3) corresponds naturally a direct sum decomposition

$$(M, \Phi) = \oplus_{j\in\kappa}(N_j, \Phi)^{s_j} \tag{4}$$

of F-crystals over k. Let c_j be the perfect, symmetric bilinear form on N_j that is the crystalline realization of b_j. As $\mathcal{V}_j$ is an O_j-module, N_j is naturally a $W(k)\otimes_{\mathbf{Z}_2} O_j$-module and c_j is $W(k) \otimes_{\mathbf{Z}_2} O_j$-linear.

4.2.1. Proposition. *We assume that for each $j \in \kappa$, the reduction modulo $2W(k)$ of c_j is a perfect, alternating form on $N_j/2N_j$. Then the flat, closed subgroup scheme J of $\mathbf{GL}_M$ is reductive.*

Proof: The formula $q_j(x) := \frac{c_j(x,x)}{2}$ defines a quadratic form on the $W(k) \otimes_{\mathbf{Z}_2} O_j$-module N_j. The closed subgroup scheme $\mathbf{SO}(N_j, q_j)$ over $W(k) \otimes_{\mathbf{Z}_2} O_j$ is isomorphic to $\mathbf{SO}^+_{2n\,W(k)\otimes_{\mathbf{Z}_2} O_j}$ (cf. [19, Prop. 3.4]) and thus it is reductive. This implies that the Zariski closure J^{der} of $J^{\mathrm{der}}_{B(k)}$ in $\mathbf{GL}_{M[\frac{1}{2}]}$ is isomorphic to $\mathbf{SO}^{+\ r}_{2n\,W(k)}$ and thus it is a reductive group scheme over $W(k)$. Let Z be the center of $\mathbf{GL}_M$. The intersection $Z \cap J^{\mathrm{der}}$ is a $\boldsymbol{\mu}_{2W(k)}$ group scheme. Let J' be the quotient of $Z \times_{W(k)} J^{\mathrm{der}}$ by a diagonal $\boldsymbol{\mu}_{2W(k)}$ closed subgroup scheme; it is a reductive group scheme (cf. [6, Vol. III, Exp. XXII, Prop. 4.3.1]) and we have a natural homomorphism $J' \to \mathbf{GL}_M$ whose fibres are closed embeddings. The homomorphism $J' \to \mathbf{GL}_M$ is a monomorphism (cf. [6, Vol. I, Exp. VI_B, Cor. 2.11]) and thus also a closed embedding (cf. [6, Vol. II, Exp. XVI, Cor. 1.5 a)]). Thus we can identify $J = J'$; thus J is a reductive group scheme. □

4.2.2. Proposition. *We assume that for each $j \in \kappa$, the reduction modulo $2W(k)$ of c_j is a perfect, alternating form on $N_j/2N_j$. Then there exists a cocharacter $\mu : \mathbf{G}_{m\,W(k)} \to J$ and a direct sum decomposition $M = F^1 \oplus F^0$ such that the following two properties hold:*

(i) *each $\beta \in \mathbf{G}_m(W(k))$ acts on F^i as the multiplication with β^{-i} (here $i \in \{0, 1\}$);*

(ii) *the k-module $F^1/2F^1$ is the kernel of the reduction ϕ modulo $2W(k)$ of Φ.*

Moreover, the normalizer of $F^1/2F^1$ in the special fibre J_k of J is a parabolic subgroup P_k of J_k and the dimension $\dim(J_k/P_k)$ is $d = \frac{rn(n-1)}{2}$.

Proof: We have a direct sum decomposition $\mathrm{Ker}(\phi) = \oplus_{j\in\kappa}(N_j/2N_j \cap \mathrm{Ker}(\phi))^{s_j}$ and each intersection $N_j/2N_j \cap \mathrm{Ker}(\phi)$ is naturally a $k \times_{\mathbf{F}_2} O_j/2O_j$-module. Thus there exists a direct summand $\tilde{F}^1_j$ of N_j that is a $W(k)\otimes_{\mathbf{Z}_2} O_j$-submodule and that lifts $N_j/2N_j \cap \mathrm{Ker}(\phi)$. Based on Proposition 4.2.1, the rest of the proof of this Proposition is the same as of [19, Prop. 6.1 and Cor. 6.1.1]. In other words, as in loc. cit. one first checks that:

(i) our hypothesis on c_j implies that we have $c_j(\tilde{F}^1_j, \tilde{F}^1_j) \in 4W(k) \otimes_{\mathbf{Z}_2} O_j$ and

(ii) property (i) and [19, Prop. 3.4] imply that there exists a direct sum decomposition $N_j = F^1_j \oplus F^0_j$ of $W(k) \otimes_{\mathbf{Z}_2} O_j$-modules such that $F^1_j/2F^1_j = \tilde{F}^1_j/2\tilde{F}^1_j$ and we have $c_j(F^1_j, F^1_j) = c_j(F^0_j, F^0_j) = 0$.

Therefore one can take $F^1 := \oplus_{j\in\kappa}(F_j^1)^{s_j}$ and $F^0 := \oplus_{j\in\kappa}(F_j^0)^{s_j}$. $\square$

4.2.3. Proposition. *We assume that for each $j \in \kappa$, the reduction modulo $2W(k)$ of c_j is a perfect, alternating form on $N_j/2N_j$. Then there exist isomorphisms $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)\tilde{\to}M^*$ of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$-modules that induce symplectic isomorphisms $(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k), \psi)\tilde{\to}(M^*, \psi_M)$.*

Proof: We refer to the $B(k)$-linear isomorphism $j_y : L_{(2)} \otimes_{\mathbf{Z}_{(2)}} B(k)\tilde{\to}M^*[\frac{1}{2}]$ of the property 4.2 (b). Let $L_y := j_y^{-1}(M^*)$. It is a $W(k)$-lattice of $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)$ such that the following three properties hold:

(i) for all $b \in \mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$ we have $b(L_y) \subseteq L_y$,

(ii) the Zariski closure of $G_{B(k)}$ in $\mathbf{GL}_{L_y}$ is a reductive group scheme $j_y^{-1}Jj_y$ over $W(k)$ (cf. Proposition 4.2.1), and

(iii) we get a perfect, alternating form $\psi : L_y \otimes_{W(k)} L_y \to W(k)$.

As in [19, Subsection 5.2], properties (i) to (iii) imply that there exists an element $g \in G^0(B(k))$ such that we have $g(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)) = L_y$. By replacing j_y with $j_y g$, we can assume that $j_y(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)) = j_y(L_y) = M^*$. Thus $j_y : L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k)\tilde{\to}M^*$ is an isomorphism of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$-modules that induces a symplectic isomorphism $(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k), \psi)\tilde{\to}(M^*, \psi_M)$. $\square$

5. Proof of the Main Theorem

In this Section we combine Sections 3 and 4 to prove the Main Theorem. We will use the notations of the lists $(\mathbf{L}_1)$ and $(\mathbf{L}_2)$ of Sections 3 and 4 (respectively). Also the notations

$$(\mathbf{L}_3)\quad \tilde{f} : (G, \mathcal{X}) \to (\mathbf{GSp}(\tilde{W}, \tilde{\psi}), \tilde{\mathcal{S}}),\ \tilde{f}' : (G', \mathcal{X}') \to (\mathbf{GSp}(\tilde{W}, \tilde{\psi}), \tilde{\mathcal{S}}),\ \tilde{L}_{(2)},\ \tilde{L},\ \tilde{\mathcal{B}},\ \tilde{\mathcal{B}}',\ v'$$

will be as in Subsections 3.3 to 3.5. Let the field k be as in Section 4. Let $\tilde{K}_2 := \mathbf{GSp}(\tilde{L}_{(2)}, \tilde{\psi})(\mathbf{Z}_2)$ and $H_2' := G'(\mathbf{Q}_2) \cap \tilde{K}_2 = G'_{\mathbf{Z}_{(2)}}(\mathbf{Z}_2)$. Let $\tilde{\mathcal{N}}$, $\tilde{\mathcal{N}}^{\mathrm{n}}$, $\tilde{\mathcal{N}}^{\mathrm{s}}$, $(\tilde{\mathcal{A}}, \tilde{\Lambda})$ (resp. $\tilde{\mathcal{N}}'$, $\tilde{\mathcal{N}}'^{\mathrm{n}}$, $\tilde{\mathcal{N}}'^{\mathrm{s}}$, $(\tilde{\mathcal{A}}', \tilde{\Lambda}')$), and $\tilde{\mathcal{M}}$ be the analogues of $\mathcal{N}$, $\mathcal{N}^{\mathrm{n}}$, $\mathcal{N}^{\mathrm{s}}$, $(\mathcal{A}, \Lambda)$, and $\mathcal{M}$ but obtained working with the hermitian orthogonal standard PEL situation $(\tilde{f}, \tilde{L}, v, \tilde{\mathcal{B}})$ (resp. $(\tilde{f}', \tilde{L}, v', \tilde{\mathcal{B}}')$) instead of with $(f, L, v, \mathcal{B})$. As the morphisms $\mathrm{Sh}(G, \mathcal{X})/H_2 \to \mathrm{Sh}(\mathbf{GSp}(\tilde{W}, \tilde{\psi}), \tilde{\mathcal{S}})_{E(G,\mathcal{X})}/\tilde{K}_2$ and $\mathrm{Sh}(G', \mathcal{X}')/H_2' \to \mathrm{Sh}(\mathbf{GSp}(\tilde{W}, \tilde{\psi}), \tilde{\mathcal{S}})_{E(G',\mathcal{X}')}/\tilde{K}_2$ are closed embeddings, we can speak about the Zariski closure $\tilde{\mathcal{N}}^{\mathrm{c}}$ of $\mathrm{Sh}(G, \mathcal{X})/H_2$ in $\tilde{\mathcal{N}}'^{\mathrm{n}}_{O_{(v)}}$. As the morphism $\tilde{\mathcal{N}}' \to \tilde{\mathcal{M}}_{O_{(v')}}$ is pro-finite, we have natural pro-finite, birational $O_{(v)}$-morphisms

$$\tilde{\mathcal{N}}^{\mathrm{n}} \to \tilde{\mathcal{N}}^{\mathrm{c}} \to \tilde{\mathcal{N}}.$$

As in [19, Subsection 4.1] we argue that the $E(G, \mathcal{X})$-scheme $\tilde{\mathcal{N}}_{E(G,\mathcal{X})} = \mathcal{N}_{E(G,\mathcal{X})}$ is regular and formally smooth and that there exists a compact, open subgroup H_0 of $G(\mathbf{A}_f^{(2)})$ such that $\mathcal{N}$, $\tilde{\mathcal{N}}$, and $\tilde{\mathcal{N}}^{\mathrm{n}}$ are pro-étale covers of the quasi-projective, normal $O_{(v)}$-schemes $\mathcal{N}/H_0$, $\tilde{\mathcal{N}}/H_0$, and $\tilde{\mathcal{N}}^{\mathrm{n}}/H_0$ (respectively) of relative dimension $d = \frac{rn(n-1)}{2}$.

5.1. Theorem. *We have* $\tilde{\mathcal{N}}^{\mathrm{n}} = \tilde{\mathcal{N}}^{\mathrm{s}} = \tilde{\mathcal{N}}^{\mathrm{c}}$.

Proof: As the $E(G,\mathcal{X})$-scheme $\tilde{\mathcal{N}}_{E(G,\mathcal{X})} = \mathcal{N}_{E(G,\mathcal{X})}$ is regular and formally smooth, to prove the Theorem it suffices to show that $\tilde{\mathcal{N}}^{\mathrm{c}}/H_0$ is a smooth $O_{(v)}$-scheme. Equivalently, it suffices to show that for each point $\tilde{y} \in \tilde{\mathcal{N}}^{\mathrm{c}}_{W(k)}(k)$, the tangent space $\mathbf{T}_{\tilde{y}}$ of $\tilde{y}$ in $\tilde{\mathcal{N}}^{\mathrm{c}}_{k}$ is a k-vector space of dimension d.

We denote also by $\tilde{y}$ the k-valued point of $\tilde{\mathcal{N}}_{W(k)}$ defined by $\tilde{y}$. Let $(\tilde{M}, \tilde{\Phi}, \psi_{\tilde{M}})$ be the principally quasi-polarized F-crystal over k of $(\tilde{A}, \lambda_{\tilde{A}}) := \tilde{y}^*((\tilde{\mathcal{A}}, \tilde{\Lambda}) \times_{\tilde{\mathcal{N}}} \tilde{\mathcal{N}}_{W(k)})$. Let

$$\tilde{L}_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2 = \oplus_{j\in\kappa} \tilde{\mathcal{V}}_j^{\tilde{s}_j} \tag{5}$$

be the direct sum decomposition that is analogous to (3) (here $\tilde{s}_j \in \mathbf{N} \setminus \{1\}$). This makes sense as (cf. end of Subsection 3.2) we can identify κ with the set of factors of $G^{\mathrm{der}}_{\mathbf{Z}_2}$ that are Weil restrictions of semisimple $\mathbf{SO}_{2n}$ group schemes. Thus we can assume that the direct factor $\mathrm{Res}_{O_j/\mathbf{Z}_2} G_{oF_{(2)}} \times_{F_{(2)}} O_j$ of $G^{\mathrm{der}}_{\mathbf{Z}_2}$ acts non-trivially on $\tilde{\mathcal{V}}_j$. Moreover, $\tilde{\mathcal{V}}_j$ is a free O_j-module of rank $2n$ and is a $G_{oF_{(2)}} \times_{F_{(2)}} O_j$-module whose fibres are simple modules. The representations $G_{oF_{(2)}} \times_{F_{(2)}} O_j \to \mathbf{GL}_{\mathcal{V}_j}$ and $G_{oF_{(2)}} \times_{F_{(2)}} O_j \to \mathbf{GL}_{\tilde{\mathcal{V}}_j}$ over O_j are isomorphic. This is so as over $W(\mathbf{F})$ they are isomorphic to $\rho_{n\,W(\mathbf{F})}$ and as their fibres are absolutely irreducible (as $n \geq 2$). We conclude that:

(i) the $G^{\mathrm{der}}_{\mathbf{Z}_2}$-modules (and thus also the $G_{\mathbf{Z}_2}$-modules) $\mathcal{V}_j$ and $\tilde{\mathcal{V}}_j$ are isomorphic.

Let $\tilde{b}_0 := \oplus_{j\in\kappa} \tilde{b}_j^{\tilde{s}_j}$ be the perfect, symmetric bilinear form on $\tilde{L}_{(2)} \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2$ that is the analogue of $b_0 = \oplus_{j\in\kappa} b_j^{s_j}$ of Subsection 4.1. Based on the property 3.5 (i), we can assume that $G'^{\mathrm{der}}_{\mathbf{Z}_2}$ fixes $\tilde{b}_0$. Let $(\tilde{M}, \tilde{\Phi}) = \oplus_{j\in\kappa} (\tilde{N}_j^{\tilde{s}_j}, \tilde{\Phi})$ be the decomposition that is the analogue of (4). Let $\tilde{c}_j$ be the perfect, symmetric bilinear form on $\tilde{N}_j$ that is the analogue of c_j of Subsection 4.1. Let $\tilde{J}$ (resp. $\tilde{J}'$) be the flat, closed subgroup scheme of $\mathbf{GL}_{\tilde{M}}$ that is the analogue of J of Subsection 4.2 and that corresponds to $\tilde{y} \in \tilde{\mathcal{N}}_{W(k)}(k)$ (resp. to the point $\tilde{y}' \in \tilde{\mathcal{N}}'_{W(k)}(k)$ defined by $\tilde{y}$).

We know that $\tilde{\mathcal{N}}'^{\mathrm{n}} = \tilde{\mathcal{N}}'^{\mathrm{s}}$ (cf. property 1.2 (i) and [19, Thm. 1.4 (c)]) and that $\tilde{J}'$ is a reductive group scheme isomorphic to $\mathbf{GSO}^{+}_{\mathbf{2n}\,W(k)}$ (cf. [19, Subsection 5.2]). Let $\tilde{c}_0 = \oplus_{j\in\kappa} \tilde{c}_j^{\tilde{s}_j}$ be the perfect, alternating form on $\tilde{M}$ that corresponds naturally to $\tilde{b}_0$. As $G'^{\mathrm{der}}_{\mathbf{Z}_2}$ fixes $\tilde{b}_0$, from [19, Thm. 5.1] applied to the point $\tilde{y}' \in \tilde{\mathcal{N}}'_{W(k)}(k)$, we get that $\tilde{c}_0$ modulo $2W(k)$ is alternating. Thus each $\tilde{c}_j$ modulo $2W(k)$ is alternating. Thus $\tilde{J}$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{\tilde{M}}$, cf. Proposition 4.2.1 applied to $\tilde{y}$. Let $\tilde{\mu} : \mathbf{G}_{m\,W(k)} \to \tilde{J}$ and $\tilde{M} = \tilde{F}^1 \oplus \tilde{F}^0$ be the analogues of the cocharacter $\mu : \mathbf{G}_{m\,W(k)} \to J$ and of the direct sum decomposition $M = F^1 \oplus F^0$ introduced in Proposition 4.2.2.

We have a natural direct sum decomposition into $W(k)$-modules

$$\mathrm{End}(\tilde{M}) = \mathrm{End}(\tilde{F}^1) \oplus \mathrm{End}(\tilde{F}^0) \oplus \mathrm{Hom}(\tilde{F}^1, \tilde{F}^0) \oplus \mathrm{Hom}(\tilde{F}^0, \tilde{F}^1)$$

as well as a modulo $2W(k)$ version of it. Let $\mathcal{O}_{\tilde{y}'}$ and $\mathcal{O}_{\tilde{y}'}^{\mathrm{big}}$ be the local rings of $\tilde{y}'$ in $\tilde{\mathcal{N}}'^{\mathrm{n}}_{W(k)}$ and in $\tilde{\mathcal{M}}_{W(k)}$ (respectively). The natural $W(k)$-homomorphism $\mathcal{O}_{\tilde{y}'}^{\mathrm{big}} \to \mathcal{O}_{\tilde{y}'}$ is surjective modulo $2W(k)$ (cf. [19, Subsection 6.7]) and thus it is surjective.

Thus the tangent space $\mathbf{T}_{\tilde{y}'}$ of $\tilde{y}'$ in $\tilde{\mathcal{N}}'^{\mathrm{n}}_{k}$ is naturally identified with the tensorization with k of the image of the Kodaira–Spencer map of the natural pull back of $\tilde{\mathcal{A}}$ to $\mathrm{Spec}(\mathcal{O}_{\tilde{y}'})$. In other words, $\mathbf{T}_{\tilde{y}'}$ is naturally identified with the intersection $\mathrm{Lie}(\tilde{J}'_k) \cap \mathrm{Hom}(\tilde{F}^1/2\tilde{F}^1, \tilde{F}^0/2\tilde{F}^0)$ (cf. [19, Fact 6.3.1 and Subsection 6.7]). Thus the tangent space $\mathbf{T}_{\tilde{y}}$ is a subspace of the intersection of $\mathrm{Lie}(\tilde{J}'_k) \cap \mathrm{Hom}(\tilde{F}^1/2\tilde{F}^1, \tilde{F}^0/2\tilde{F}^0)$ with the centralizer of $\tilde{\mathcal{B}}^{\mathrm{opp}} \otimes_{\mathbf{Z}_{(2)}} \mathbf{F}$ in $\mathrm{End}(\tilde{M}/2\tilde{M})$. By applying Proposition 4.2.3 to $\tilde{y}$ and based on the property 1.2 (iii), we get that the identity component of the centralizer of $\tilde{\mathcal{B}}^{\mathrm{opp}} \otimes_{\mathbf{Z}_{(2)}} \mathbf{F}$ in $\tilde{J}'$ is $\tilde{J}$. Thus $\mathbf{T}_{\tilde{y}}$ is a subspace of the intersection $\mathrm{Lie}(\tilde{J}_k) \cap \mathrm{Hom}(\tilde{F}^1/2\tilde{F}^1, \tilde{F}^0/2\tilde{F}^0)$. Let $\tilde{P}_k$ be the parabolic subgroup of $\tilde{J}_k$ that is the normalizer of $\tilde{F}^1/2\tilde{F}^1$ in $\tilde{J}_k$, cf. Proposition 4.2.2 applied to $\tilde{y}$. As $\tilde{\mu}$ is a cocharacter of $\tilde{J}$, we have an identity $\mathrm{Lie}(\tilde{J}_k) = \mathrm{Lie}(\tilde{P}_k) \oplus (\mathrm{Lie}(\tilde{J}_k) \cap \mathrm{Hom}(\tilde{F}^1/2\tilde{F}^1, \tilde{F}^0/2\tilde{F}^0))$. Thus $\dim_k(\mathbf{T}_{\tilde{y}}) \leq \dim_k(\mathrm{Lie}(\tilde{J}_k)/\mathrm{Lie}(\tilde{P}_k)) = \dim(\tilde{J}_k/\tilde{P}_k)$ and therefore (cf. Proposition 4.2.2 applied to $\tilde{y}$) we have $\dim_k(\mathbf{T}_{\tilde{y}}) \leq d$. As we obviously have $\dim_k(\mathbf{T}_{\tilde{y}}) \geq d$, we get that $\dim_k(\mathbf{T}_{\tilde{y}}) = d$. □

5.2. Proposition. *The natural identification of $\tilde{\mathcal{N}}^{\mathrm{n}}_{E(G,\mathcal{X})}$ with $\mathcal{N}^{\mathrm{n}}_{E(G,\mathcal{X})}$ extends uniquely to an $O_{(v)}$-morphism $\Xi : \tilde{\mathcal{N}}^{\mathrm{n}} \to \mathcal{N}^{\mathrm{n}}$.*

Proof: To ease notations, let $\tilde{Y} := \tilde{\mathcal{N}}^{\mathrm{n}}_{W(k)}$. Let $(\mathfrak{D}, \lambda_{\mathfrak{D}})$ and $(\tilde{\mathfrak{D}}, \lambda_{\tilde{\mathfrak{D}}})$ be the principally quasi-polarized 2-divisible groups of $(\mathcal{A}, \Lambda)_{\mathcal{N}^{\mathrm{n}}_{W(k)}}$ and $(\tilde{\mathcal{A}}, \tilde{\Lambda})_{\tilde{Y}}$ (respectively). To the decompositions (3) and (5) correspond naturally decompositions

$$\mathfrak{D} = \oplus_{j\in\kappa}\mathfrak{D}_j^{s_j} \text{ and } \tilde{\mathfrak{D}} = \oplus_{j\in\kappa}\tilde{\mathfrak{D}}_j^{\tilde{s}_j}$$

(respectively) into 2-divisible groups. The fact that the $G_{\mathbf{Z}_2}$-modules $\mathcal{V}_j$ and $\tilde{\mathcal{V}}_j$ are isomorphic (see Theorem 5.1 (i)) can be encoded in the existence of a suitable $\mathbf{Z}_2$-endomorphism between the abelian schemes $\mathcal{A}_{\mathcal{N}_{B(k)}}$ and $\tilde{\mathcal{A}}_{\tilde{\mathcal{N}}_{B(k)}}$ over $\mathcal{N}_{B(k)} = \tilde{Y}_{B(k)}$; this $\mathbf{Z}_2$-endomorphism allows us to identify naturally $\mathfrak{D}_{jB(k)}$ with $\tilde{\mathfrak{D}}_{jB(k)}$ as 2-divisible groups over $\mathcal{N}^{\mathrm{n}}_{B(k)} = \tilde{Y}_{B(k)}$. Thus we can speak about the 2-divisible group $\tilde{\mathfrak{E}} := \oplus_{j\in\kappa}\tilde{\mathfrak{D}}_j^{s_j}$ over $\tilde{Y}$. Let $\lambda_{\tilde{\mathfrak{E}}B(k)} := \lambda_{\mathfrak{D}B(k)}$; it is a principal quasi-polarization of $\tilde{\mathfrak{E}}_{B(k)}$. As the $O_{(v)}$-scheme $\tilde{Y}$ is flat and normal, a theorem of Tate (see [15, Thm. 4]) implies that $\lambda_{\tilde{\mathfrak{E}}B(k)}$ extends uniquely to a principal quasi-polarization $\lambda_{\tilde{\mathfrak{E}}}$ of $\tilde{\mathfrak{E}}$.

Let $\tilde{O}$ be a local ring of $\tilde{Y}$ that is a discrete valuation ring of mixed characteristic $(0, 2)$. Let $\tilde{K}$ be the field of fractions of $\tilde{O}$. We also view the natural $E(G, \mathcal{X})$-morphism $\tilde{z} : \mathrm{Spec}(\tilde{K}) \to \tilde{Y}_{B(k)}$ as a $\tilde{K}$-valued point $\tilde{z}$ of $\mathcal{N}^{\mathrm{n}}_{B(k)}$. Thus we can speak about the abelian variety $\tilde{A}_{\tilde{K}} := \tilde{z}^*(\mathcal{A}_{\mathcal{N}^{\mathrm{n}}_{B(k)}})$ over $\tilde{K}$. As $\tilde{A}_{\tilde{K}}$ has a level-l structure for each odd natural number l, it extends to an abelian scheme $\tilde{A}_{\tilde{O}}$ over $\tilde{O}$ (cf. the Néron–Shafarevich–Ogg criterion of good reduction; see [1, Ch. 7, 7.4, Thm. 5]). It is easy to see that each

principal quasi-polarization of $\tilde{A}_{\tilde{K}}$ extends to a principal polarization of $\tilde{A}_{\tilde{O}}$ (at the level of isomorphisms between abelian scheme over $\tilde{O}$ this follows from [1, Ch. 7, 7.5, Prop. 3]). We conclude that there exists an open subscheme $\tilde{U}$ of $\tilde{Y}$ such that:

(i) we have $\tilde{Y}_{B(k)} \subseteq \tilde{U}$ and the codimension of $\tilde{U}$ in $\tilde{Y}$ is at least 2;

(ii) the principally polarized abelian scheme $(\mathcal{A}, \Lambda)_{\tilde{Y}_{B(k)}} = (\mathcal{A}, \Lambda)_{\mathcal{N}^{\mathrm{n}}_{B(k)}}$ extends to a principally polarized abelian scheme $(\tilde{A}_{\tilde{U}}, \lambda_{\tilde{A}_{\tilde{U}}})$ over $\tilde{U}$.

Tate theorem also implies that the principally quasi-polarized 2-divisible group of $(\tilde{A}_{\tilde{U}}, \lambda_{\tilde{A}_{\tilde{U}}})$ is $(\tilde{\mathfrak{E}}, \lambda_{\tilde{\mathfrak{E}}})_{\tilde{U}}$. As the scheme $\tilde{Y}$ is regular, from [18, proof of Prop. 4.1] we get that the principally polarized abelian scheme $(\tilde{A}_{\tilde{U}}, \lambda_{\tilde{A}_{\tilde{U}}})$ over $\tilde{U}$ extends to a principally polarized abelian scheme $(\tilde{A}_{\tilde{Y}}, \lambda_{\tilde{A}_{\tilde{Y}}})$ over $\tilde{Y}$ whose principally quasi-polarized 2-divisible group is $(\tilde{\mathfrak{E}}, \lambda_{\tilde{\mathfrak{E}}})$. The natural level-$l$ symplectic similitude structures of $(\mathcal{A}, \Lambda)_{\tilde{Y}_{B(k)}}$ extend naturally to level-l symplectic similitude structures of $(\tilde{A}_{\tilde{Y}}, \lambda_{\tilde{A}_{\tilde{Y}}})$. Thus the natural $E(G, \mathcal{X})$-morphism $\tilde{Y}_{B(k)} \to \mathcal{M}_{B(k)}$ extends uniquely to a morphism $\tilde{Y} \to \mathcal{M}$. This last morphism factors through a $W(k)$-morphism $\Xi_{W(k)} : \tilde{Y} \to \mathcal{N}^{\mathrm{n}}_{W(k)}$ which is the pull back to $W(k)$ of the searched for $O_{(v)}$-morphism $\Xi : \tilde{\mathcal{N}}^{\mathrm{n}} \to \mathcal{N}^{\mathrm{n}}$. Obviously Ξ is unique. $\square$

5.2.1. Simple facts. (a) The existence of Ξ is also a direct consequence of [18, Thm. 1.3] and of the fact (see [17, Example 3.2.9 and Cor. 3.4.1]) that the $O_{(v)}$-scheme $\mathcal{N}^{\mathrm{n}}$ has the extension property (with respect to $O_{(v)}$-schemes which are healthy regular) defined in [17, Def. 3.2.3 3)].

(b) The $W(k)$-morphism $\Xi : \tilde{\mathcal{N}}^{\mathrm{n}} \to \mathcal{N}^{\mathrm{n}}$ is the pull back of an $O_{(v)}$-morphism $\Xi_{H_0} : \tilde{\mathcal{N}}^{\mathrm{n}}/H_0 \to \mathcal{N}^{\mathrm{n}}/H_0$ whose generic fibre is a natural identification (and thus an isomorphism).

(c) As $\tilde{\mathcal{N}}'^{\mathrm{n}} = \tilde{\mathcal{N}}'^{\mathrm{s}}$, the k-scheme $\tilde{\mathcal{N}}'^{\mathrm{n}}_k$ is reduced. Thus the natural k-morphism $\tilde{\mathcal{N}}'^{\mathrm{n}}_k \to \tilde{\mathcal{M}}_k$ induces k-epimorphisms at the level of complete, local rings of residue field k (i.e., it is a formally closed embedding at all k-valued points), cf. [19, Subsection 6.7]. Thus as $\tilde{\mathcal{N}}^{\mathrm{n}} = \tilde{\mathcal{N}}^{\mathrm{c}}$ is a closed subscheme of $\tilde{\mathcal{N}}'^{\mathrm{n}}_{O_{(v)}}$, the principally quasi-polarized 2-divisible group $(\tilde{\mathfrak{E}}, \lambda_{\tilde{\mathfrak{E}}}) \times_{W(k)} k$ is a versal deformation at each k-valued point of $\tilde{Y}_k = \tilde{\mathcal{N}}^{\mathrm{n}}_k$.

5.3. Lemma. *The $O_{(v)}$-morphism $\Xi_{H_0} : \tilde{\mathcal{N}}^{\mathrm{n}}/H_0 \to \mathcal{N}^{\mathrm{n}}/H_0$ is projective.*

Proof: As Ξ_{H_0} is quasi-projective and its generic fibre is an isomorphism, it suffices to show that for each discrete valuation ring O that is a faithfully flat $O_{(v)}$-algebra, every $O_{(v)}$-morphism $\mathrm{Spec}(O) \to \mathcal{N}^{\mathrm{n}}/H_0$ factors uniquely through $\tilde{\mathcal{N}}^{\mathrm{n}}/H_0$. To check this, we can assume that O is strictly henselian and it suffices to show that each $O_{(v)}$-morphism $h_O : \mathrm{Spec}(O) \to \mathcal{N}^{\mathrm{n}}$ factors uniquely through $\tilde{\mathcal{N}}^{\mathrm{n}}$. Let $K := O[\frac{1}{2}]$. The principally polarized abelian variety $(\tilde{\mathcal{A}}, \tilde{\Lambda})_K$ over K extends to a principally polarized abelian scheme over O, cf. the Néron–Shafarevich–Ogg criterion of good reduction and [1, Ch. 7, 7.5, Prop. 3]. This implies that the generic fibre of h_O extends uniquely to

a morphism $l_O : \mathrm{Spec}(O) \to \tilde{\mathcal{M}}$ which factors through $\tilde{\mathcal{N}}^{\mathrm{n}}$. Thus $h_O : \mathrm{Spec}(O) \to \mathcal{N}^{\mathrm{n}}$ factors uniquely through $\tilde{\mathcal{N}}^{\mathrm{n}}$. $\square$

5.4. End of the proof of 1.3. Due to the versality part of Subsubsection 5.2.1 (c), the pull back to $\mathrm{Spec}(k)$ of the $O_{(v)}$-morphism $\Xi : \tilde{\mathcal{N}}^{\mathrm{n}} \to \mathcal{N}^{\mathrm{n}}$ induces k-epimorphisms at the level or complete, local rings of residue field k (i.e., it is a formally closed embedding at all k-valued points). This implies that the fibres of the $O_{(v)}$-morphism Ξ_{H_0} are finite. Therefore Ξ_{H_0} is a quasi-finite morphism. From this and Lemma 5.3 we get that Ξ_{H_0} is a finite, birational $O_{(v)}$-morphism. As the quasi-projective $O_{(v)}$-scheme $\mathcal{N}^{\mathrm{n}}/H_0$ is normal, we conclude that Ξ_{H_0} is an isomorphism. Thus the $O_{(v)}$-morphism $\Xi : \tilde{\mathcal{N}}^{\mathrm{n}} \to \mathcal{N}^{\mathrm{n}}$ is an isomorphism. Thus the $O_{(v)}$-scheme $\mathcal{N}^{\mathrm{n}}$ is regular and formally smooth, cf. Theorem 5.1; this implies that $\mathcal{N}^{\mathrm{n}} = \mathcal{N}^{\mathrm{s}}$. This ends the proof of the Main Theorem.

5.5. Corollary. *Let $y \in \mathcal{N}_{W(k)}(k)$. Let (M, Φ, ψ_M) be as in the beginning of Section 4. Then there exist isomorphisms $L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k) \tilde{\to} M^*$ of $\mathcal{B} \otimes_{\mathbf{Z}_{(2)}} W(k)$-modules that induce symplectic isomorphisms $(L_{(2)} \otimes_{\mathbf{Z}_{(2)}} W(k), \psi) \tilde{\to} (M^*, \psi_M)$.*

Proof: As $\mathcal{N}^{\mathrm{n}} = \mathcal{N}^{\mathrm{s}}$ (cf. Main Theorem), there exists a lift $z \in \mathcal{N}_{W(k)}(W(k))$ of y. Let F be the Hodge filtration of M defined by $z^*(\mathcal{A}_{\mathcal{N}_{W(k)}})$. The decomposition (4) extends to a decomposition

$$(M, F, \Phi) = \oplus_{j \in \kappa} (N_j, F_j, \Phi)^{s_j}$$

of filtered F-crystals over k. For $x,\, u \in N_j$ we have $c_j(\Phi(x), \Phi(u)) = 2\sigma(c_j(x, u))$, where σ is the Frobenius automorphism of $W(k)$. Moreover if $v \in F_j$, then $c_j(v, v) = 0$ and $\Phi(v) \in 2N_j$; thus $c_j(\frac{\Phi(v)}{2}, \frac{\Phi(v)}{2}) = 0$. As $N_j = \Phi(N_j) + \frac{1}{2}\Phi(F_j)$, each $x \in N_j$ is a sum $\frac{1}{2}\Phi(v) + \Phi(u)$, with $u \in N_j$ and $v \in F_j$; thus

$$c_j(x, x) = 2c_j(\Phi(u), \frac{1}{2}\Phi(v)) + c_j(\Phi(u), \Phi(u)) = 2c_j(\Phi(u), \frac{1}{2}\Phi(v)) + 2\sigma(c_j(u, u)) \in 2W(k).$$

Thus c_j modulo $2W(k)$ is alternating. Therefore the Corollary follows from Proposition 4.2.3. $\square$

5.6. Remark. The proofs of [19, Thm. 1.4 (b) and Cor. 6.8] can be easily adapted to show that the reduced schemes of $\mathcal{N}_k$ and $\mathcal{N}_k^{\mathrm{n}}$ are regular and formally smooth over k.

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